UNIT 9 INTEGRATION OF ELEMENTARY FUNCTIONS

Structure

9.1 INTRODUCTION

In Unit 6, we developed techniques of differentiation which enable us to differentiate almost any function with comparative ease. Although, integration is the reverse process of differentiation as we have seen in Unit 8, yet integration is much harder to carry out. Recall that in the case of differentiation, if a function is an expression involving elementary functions (such as x^r, sin x, e^x, ...) then, so is its derivative. Although many integration problems also have this characteristic, certain ones do not. However, there are some elementary functions (e.g. e^{x^2}) for which an integral cannot be expressed in terms of elementary functions. Even where this is possible, the techniques for finding these integrals are often complicated. For this reason, we must be prepared with a broad range of techniques in order to cope with the problem of calculating integrals.

In this unit we will develop two general techniques, namely, integration by substitution and integration by parts for calculating both indefinite and definite integrals. We will also discuss their application for the integration of various classes of elementary and trigonometric functions.

Objectives

After reading this unit you should be able to :

- compute integrals of functions using standard integrals,
- use the method of substitution for integration,
- use the method of integration by parts for integration,
- compute integrals of various elementary and trigonometric functions.

9.2 STANDARD INTEGRALS

In many cases a function is at once recognised as the derivative of some other function and thus can be integrated easily. Such integrals are known as **Standard Integrals.** We now list such standard integrals for ready reference in Table 1. We shall be making use of these integrals every now and then while evaluating many more integrals. Before we give these integrals let us mention that throughout the discussion in this unit we shall be denoting the constant of integration by c.

It is important to note that when $n \neq -1$, the integral or x^n is obtained on increasing the index n by 1 and dividing by the increased index n+1. Thus, for example,

$$\int x^{1/2}dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}x^{3/2},$$

and

$$\int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} + c = -\frac{1}{x} + c.,$$

Table 1

$\int x^n\,dx$	=	$\frac{x^{n+1}}{n+1}$, $(n \neq -1)$	$\int \operatorname{cosec}^2 x\,dx$	=	$-\cot x$
$\int \frac{1}{x}\,dx$	=	$\ln x$	$\int \sec x \tan x\,dx$	=	$\sec x$
$\int e^x\,dx$	=	e^x	$\int \operatorname{cosec} x \cot x\,dx$	=	$-\operatorname{cosec} x$
$\int a^x\,dx$	=	$\frac{a^x}{\ln a}$	$\int \left[\frac{1}{\sqrt{1-x^2}}\right] dx$	=	$\sin^{-1}x$
$\int \sin x\,dx$	=	$-\cos x$	$\int \left[\frac{1}{(1+x^2)}\right] dx$	=	$\tan^{-1}x$
$\int \cos x\,dx$	=	$\sin x$	$\int \left[\frac{1}{x\sqrt{(x^2-1)}}\right] dx$	=	$\sec^{-1}x$
$\int \sec^2 x\,dx$	=	$\tan x$			

We shall also be making repeated use of the following two properties of indefinite integral (ref. Unit 8).

$$\int \{a\, f(x)\}\,dx = a\int f(x)\,dx$$

$$\int \{f(x) \pm g(x)\}\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Remember that these results also hold for the sum (or difference) of a finite number of functions.

Let us now do few examples by making use of the standard integrals and the above two properties.

Example 1 : Integrate 2sin x with respect to x.

Solution : $\int 2\sin x\,dx = 2\int \sin x\,dx = -2\cos x + c.$

Example 2 : Integrate with respect to x,

i) $\sqrt{x} + 2^x$ ii) $\cos^2(x/2)$

Solution : i) $\int (\sqrt{x} + 2^x)\,dx = \int \sqrt{x}\,dx + \int 2^x\,dx$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{3}{2}} + \frac{2^x}{\ln 2} + c$$

$$= \frac{2x^{3/2}}{3} + \frac{2^x}{\ln 2} + c$$

ii) $\int \cos^2(x/2)\,dx = \int \frac{1+\cos x}{2}\,dx$ $(\because \cos 2x = 2\cos^2 x - 1)$

$$= \int \frac{1}{2}\,dx + \int \frac{1}{2}\cos x\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos x\,dx$$

$$= \frac{1}{2}x + \frac{1}{2}\sin x + c.$$

Let us now take up a few examples of definite integrals.

Example 3 : Evaluate $\int_0^{\pi/4} \tan^2 x \, dx$.

Solution : $\int_0^{\pi/4} \tan^2 x \, dx = \int_0^{\pi/4} (\sec^2 x - 1)\, dx \; (\because 1 + \tan^2 x = \sec^2 x)$

$$= \int_0^{\pi/4} \sec^2 x \, dx - \int_0^{\pi/4} dx$$

$$= \Big[\tan x\Big]_0^{\pi/4} - [x]_0^{\pi/4}$$

$$= \tan(\pi/4) - \tan 0 - (\pi/4 - 0)$$

$$= 1 - 0 - \pi/4 + 0$$

Therefore, $\int_0^{\pi/4} \tan^2 x \, dx = 1 - \pi/4$.

Example 4 : Evaluate $\int_0^{\pi/4} \frac{1-x^2}{1+x^2}\, dx$.

Solution : We first write $\frac{1-x^2}{1+x^2} = \frac{2-(1+x^2)}{1+x^2} = \frac{2}{1+x^2} - 1$.

Now, integrating both sides of the above equation, we get

$$\int_0^{\pi/4} \frac{1-x^2}{1+x^2}\, dx = \int_0^{\pi/4} \left(\frac{2}{1+x^2} - 1\right) dx$$

$$= \int_0^{\pi/4} \frac{2}{1+x^2}\, dx - \int_0^{\pi/4} dx$$

$$= 2 \int_0^{\pi/4} \frac{dx}{(1+x^2)} dx - [x]_0^{\pi/4}$$

$$= 2 \Big[\tan^{-1} x\Big]_0^{\pi/4} - \frac{\pi}{4}$$

$$= 2\Big[\tan^{-1}(\pi/4) - \tan^{-1} 0\Big] - \frac{\pi}{4}$$

so that, $\int_0^{\pi/4} \frac{1-x^2}{1+x^2}\, dx = 2 - \frac{\pi}{4}$.

You must have noticed that in the two examples above, our effort has been to express the given integrand as a combination of some standard integrals listed in Table 1. You can easily solve these exercises now, by using the same strategy.

E1) Integrate the following with respect to x.

a) $\frac{\sin x}{\cos^2 x}$ b) $\cot^2 x$

c) $\frac{a+bx+cx^2}{x^2}$ d) $\sqrt{x}\,(ax^2 + bx + d)$

e) $\frac{x^4+1}{x^2+1}$ f) $x^a + a^x$

E2) Evaluate the following integrals :

a) $\int_0^2 \frac{1}{x}\, dx$ b) $\int_0^{\pi/2} \frac{\sin^2\theta}{1+\cos\theta}\, d\theta$ c) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

From the examples and exercises above you may have realised the need for developing techniques of integration. Functions like sin x, cos x, $\sec^2 x$, $\operatorname{cosec}^2 x$ occur as integrals of simple trigonometric functions (namely, −cos x, sin x, tan x and cot x respectively). But there are some trigonometric functions like tan x, cot x, sec x, cosec x, etc. which do not occur in Table 1. In the next section, we will develop techniques of integration of such and other classes of functions. In fact, the main use of these techniques will be to convert functions belonging to each class into one of the standard integrals and then evaluate the same by using Table 1.

9.3 METHODS OF INTEGRATION

We have seen in Section 9.2 that the decomposition of an integrand into the sum of a number of integrands with known integrals, is itself an important method of integration.

We now give two general methods of integration, namely,

i) integration by substitution,

ii) integration by parts.

The method of substitution (also referred to as change of variables) consists in expressing the integral $\int f(x)\,dx$ in terms of another simpler integral, $\int F(t)\,dt$, say, where the variables x and t are connected by some suitable relation $x = \phi(t)$.

The method of integration by parts enables one to express the given integral of a product of two functions in terms of another, whose integration may be simpler. We now discuss these methods one by one.

9.3.1 Integration by Substitution

We have noted earlier, in Table 1, that every differentiation formula can be turned into a corresponding integration formula. This is true even for the chain-rule and the resulting formula is called **integration** by **substitution.** This is often used to transform a complicated integral into a simpler one. To be more explicit, consider the following situation.

Let f(x) be any given function and

$$F(x) = \int f(x)\,dx \quad \left(\text{therefore, } \frac{dF}{dx} = f(x)\right) \ldots\ldots\ldots\ldots\ldots (1)$$

Suppose further that $x = \phi(t)$ is a single-valued function.
Then by, the chain rule,

$$\frac{dF(x)}{dt} = \frac{dF(x)}{dx} \cdot \frac{d\phi(t)}{dt} = f(x)\frac{d\phi}{dt} \quad (\text{ by (1) })$$

$$\text{Therefore, } \frac{dF(x)}{dt} = f(\phi(t))\ \phi'(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where dash denotes differentiation with respect to t. Integrating both sides of (2) with respect to t we get

$$\int \frac{dF(x)}{dt}\,dt = \int f(\phi(t))\ \phi'(t)\,dt$$

$$\text{or} \qquad F(x) = \int f(\phi(t))\ \phi'(t)\,dt$$

$$\text{or} \qquad \int f(x)\,dx = \int f(\phi(t))\ \phi\ (t)\,dt$$

Let us now illustrate this technique with examples.

Example 5 : Find $\int (x^2+1)^3\, 2x\,dx$.

Solution : Let $t=x^2+1$. Then $dt = 2x\,dx$

Therefore, $\int (x^3+1)^3 \,.\, 2x\, dx = \int t^3\, dt = \frac{1}{4} t^4 + c$

Thus, $\int (x^2+1)^3\, 2x\, dx = \frac{1}{4} (x^2+1)^4 + c$ since $t = x^2+1$.

Example 6 : Find $\int 2x\, e^{x^2} dx$.

Solution : Let $t = x^2$. Then $dt = 2x\, dx$

$$\therefore \int 2x\, e^{x^2} dx = \int e^t\, dt = e^t + c$$

or $\int 2x\, e^{x^2} dx = e^{x^2} dx + c$ $(\because t = x^2)$.

From Examples 5 and 6 we deduce that the method of integration by substitution involves the following steps :

Step 1 : Define a new variable $t = \phi(x)$, where $\phi(x)$ is chosen in such a way that, when a given integral $\int f(x)\, dx$ is written in terms of t, the integrand becomes simpler.

Step 2 : Transform the integral with respect to x into an integral with respect to t by replacing $\phi(x)$ everywhere by t and $\phi'(x)dx$ by dt.

Step 3 : Integrate the resulting function of t.

Step 4 : Rewrite the answer in terms of x by replacing t by $\phi(x)$.

Let us now consider another example which illustrates these various steps.

Example 7 : Obtain $\int \frac{(\ln x)^2}{x} dx$.

Solution : Step 1 : Let $t = \ln x$. Then $dt = \frac{1}{x} dx$.

Step 2 : $\int \frac{(\ln x)^2}{x} dx = \int (\ln x)^2 \,.\, \frac{1}{x} dx$

$$= \int t^2\, dt \qquad \text{..............} \quad (3)$$

Step 3 : $\int t^2\, dt = \frac{t^3}{3} + c$ .... (4)

Step 4 : $\frac{t^3}{3} + c = \frac{(\ln x)^3}{3} + c$ .... (5)

since, $t = \ln x$.

Therefore, from (3) and (5) we get,

$$\int \frac{(\ln x)^2}{x} dx = \frac{(\ln x)^3}{3} + c$$

Note : Making a suitable substitution is a skill which you can develop through practice. Basically, we try to look for a composition of the form $f(\phi(t))\, \phi'(t)$, where $f(\phi(t))$ is a function whose integral is known to us and $\phi'(t)$ appears in the integrand. Sometimes it may happen that instead of $\phi'(t)$ a constant multiple of $\phi'(t)$ may appear in the integrals.

Let us now illustrate this point with the help of an example.

Example 8 : Find $\int x^3\, e^{x^4} dx$.

Solution : Step 1 : Let $t = x^4$, then $dt = 4x^3\, dx$.

Thus, the needed factor 4 can be introduced in the integrand and

$\int x^3 e^{x^4} dx = \int \frac{1}{4}\, 4x^3\, e^{x^4} dx$ (multiply and divide by 4)

$$= \frac{1}{4} \int e^{x^4}\, 4x^3\, dx \qquad \text{....................} \quad (6)$$

Step 2 : $\int x^3 e^{x^4} dx = \frac{1}{4}\int e^t dt$ .... (7)

Step 3 : $\frac{1}{4}\int e^t dt = \frac{1}{4} e^t + c$ .... (8)

Step 4 : Since, $t = x^4$, from (7) and (8) we get,

$$\int x^3 e^{x^4} dx = \frac{1}{4} e^{x^4} + c$$

You may now try the following exercise :

E3) obtain $\int f(x)\, dx$, where

a) $f(x) = (3x-2)^3$ b) $f(x) = \sqrt{3-2x}$

c) $f(x) = e^{2x+3}$ d) $f(x) = \sin mx$.

Some Typical Examples of Substitution

We now consider the integrals $\int f(x)\, dx$, where the integrand f(x) is in some typical form and the integral can be obtained easily by the method of substitution. Various forms of integrals considered are as follows:

(A) $\int f(ax+b)\, dx$

To integrate $f(ax+b)$, put $ax+b = t$. Therefore, $a\, dx = dt$ or $dx = \frac{1}{a} dt$

Thus, $\int f(ax+b)\, dx = \frac{1}{a}\int f(t)\, dt$

which can be evaluated, once the right hand side is known.

For example, to find $\int \cos(ax+b)dx$, we put $ax+b = t$ and $adx = dt$, or $dx = \frac{1}{a}$ dt.

Then, $\int \cos(ax+b) = \frac{1}{a}\int \cos t\, dt = \frac{1}{a} \sin t + c$

or $\int \cos(ax+b)dx = \frac{1}{a} \sin(ax+b) + c$

Similarly, we have the following results.

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c,\ n \neq -1$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln(ax+b) + c$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{(ax+b)} + c$$

$$\int \sec^2(ax+b)dx = \frac{1}{a} \tan(ax+b) + c \text{ etc.}$$

You can make direct use of the above results in solving exercises.

(B) $\int f(x^n)x^{n-1}\, dx$

To integrate $f(x^n)\, x^{n-1}$ we let $x^n = t$. Then $nx^{n-1}\, dx = dt$

and $\int f(x^n)x^{n-1}\, dx = \frac{1}{n}\int f(t)\, dt$

which can be found out once the right hand side is known.

For example, to find $\int x^2 \sin x^3 dx$, put $x^3 = t$. Then $3x^2dx = dt$, that is, $x^2dx = \frac{1}{3}$ dt

Then, $\int x^2 \sin x^3\, dx = \frac{1}{3} \int \sin t\, dt = -\frac{1}{3} \cos t + c$

$$= -\frac{1}{3} \cos x^3 + c$$

(C) $\int f(x)^n f'(x)\, dx, \quad n \neq -1$

Putting $f(x) = t$, we see that $f'(x)dx = dt$ and

$$\int \{f(x)\}^n f'(x)\, dx = \int t^n\, dt = \frac{t^{n+1}}{n+1} + c$$

$$= \int \frac{\{f(x)\}^{n+1}}{n+1} + c$$

For example, $\int \cos^2 x \sin x\, dx = -\int t^2\, dt$, where, $t = \cos x$ (and hence $-dt = \sin x\, dx$).

Therefore, $\int \cos^2 x \sin x\, dx = -\frac{1}{3} t^3 + c = -\frac{1}{3} \cos^3 x + c$

(D) $\int \frac{f'(x)}{f(x)}\, dx$

Putting $f(x) = t$, we have $f'(x)dx = dt$

and $\int \frac{f'(x)}{f(x)} = \int \frac{dt}{t} = \ln t + c = \ln f(x) + c$

i.e., the integral of a fraction in which the numerator is the differential coefficient of the denominator, is equal to the logarithm of the denominator (plus a constant). For example, applying this result, we have

$$\int \frac{\sin x}{\cos x}\, dx = -\int \frac{(-\sin x)}{\cos x}\, dx = c - \ln \cos x,$$

since $f(x) = \cos x$ in this case.

Therefore, $\int \tan x\, dx = c - \ln \cos x$

Similarly, you can obtain the following integrals:

$$\int \cot x\, dx = \ln \sin x + c$$

$$\int \sec x\, dx = \ln(\sec x + \tan x) + c$$

$$\int \operatorname{cosec} x\, dx = \ln \tan(x/2) + c$$

Remember that logarithm of a quantity is defined only when the quantity is positive. Thus, while making use of these formulas make sure that the integrand to be integrated is positive in the domain under consideration.

(E) $\int f(a^2 \pm x^2)\, dx$

Under this category we now give some results obtained by putting $x = at$, and hence $dx = a\, dt$.

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2-x^2}\, dx = \frac{1}{2} x \sqrt{a^2-x^2} + \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c.$$

We usually use all the integrals given under (A)–(E) directly whenever required without actually proving them. Using these formulas, you may now try the following exercise:

E4) Integrate the following v ·h respect to x.

a) $x \sec^2 x^2$ b) $\dfrac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}$ c) $\dfrac{(1+\ln x)^3}{x}$

d) $\dfrac{\operatorname{cosec}^2 x}{(1+\cot x)}$ e) $\dfrac{x}{\sqrt{1-2x^4}}$

Sometimes it may happen that two or more substitutions have to be used in succession. We now illustrate this point with the help of the following example.

Example 9 : Obtain $\int \dfrac{x^2 \tan^{-1}x^3}{(1+x^6)}\, dx$.

Solution : Step 1 : Put $x^3 = t$. Then, $3x^2 dx = dt$

Therefore, $\int \dfrac{x^2\tan^{-1}x^3}{(1+x^6)}\, dx = \dfrac{1}{3}\int \dfrac{\tan^{-1}t}{1+t^2}\, dt$ .... (9)

Step 2 : Put $\tan^{-1}t = u$, so that $\dfrac{1}{1+t^2}\, dt = du$

Then, the right hand side of (9) becomes

$\dfrac{1}{3}\int u\, du = \dfrac{1}{3}\cdot\dfrac{1}{2}u^2 + c$ .... (10)

Step 3 : From (9) and (10) we have

$$\int \frac{x^2\tan^{-1}x^3}{(1+x^6)}\, dx = \frac{1}{6}u^2 + c = \frac{1}{6}(\tan^{-1}t)^2 + c = \frac{1}{6}(\tan^{-1}x^3)^2 + c$$

for $u = \tan^{-1}t$ and $t = x^3$.

We now carry over the application of the method of substitution for solving definite integrals.

Evaluation of definite integrals by substitution

Suppose that an integral of the form $\int_a^b f(\phi(x))\,\phi'(x)\, dx$ is subjected to the substitution of $t = \phi(x)$, so that

$\int f(\phi(x))\,\phi'(x)\,dx$ becomes $\int f(t)\, dt$; then the limits of integration also changes accordingly. In this case, for $x = a$, $t = \phi(a)$ and for $x = b$, $t = \phi(b)$ and

$$\int_a^b f(\phi(x))\,\phi'(x)\, dx = \int_{\phi(a)}^{\phi(b)} f(t)\, dt.$$

Now, if $F(x)$ is an integral of $f(x)$, then

$$\frac{d}{dx}[\,F(\phi(x))\,] = F'(\phi(x))\,\phi'(x)$$
$$= f(\phi(x))\,\phi'(x) \quad (\text{since } F(x) = \int f(x)\, dx)$$

$$\text{and } \int f(\phi(x))\,\phi'(x)\, dx = F(\phi(x))\Big|_a^b$$
$$= F(\phi(b)) - F(\phi(a)).$$

Thus, when the variable is changed from x to t, the new limits are the values of t which correspond to the values a and b of x. Various underlying steps in this method are elaborated through the next example.

Example 10 : Evaluate $\int_0^8 \dfrac{x}{\sqrt{x+1}}\, dx$.

Solution : Step 1 : Put $t = \sqrt{x+1}$. Therefore $t^2 = x+1$ .... (11)

and $x = t^2-1$ so, $dx = 2t\,dt$.

From (11) when x=0, t=1 and when x=8, t=3.

$$\text{Thus, } \int_0^8 \frac{x}{\sqrt{x+1}}\,dx = \int_1^3 \frac{t^2-1}{t}\,2t\,dt$$

$$\text{or } \int_0^8 \frac{x}{\sqrt{x+1}}\,dx = 2\int_1^3 (t^2-1)\,dt \quad \text{........ (12)}$$

$$\textbf{Step 2 : } 2\int_1^3 (t^2-1)\,dt = 2\left[\frac{t^3}{3} - t\right]_1^3 = 2\left(\frac{20}{3}\right) = \frac{40}{3} \quad \text{........ (13)}$$

Step 3 : From (12) and (13) we have

$$\int_0^8 \frac{x}{\sqrt{x+1}}\,dx = \frac{40}{3}$$

And now an exercise for you.

E5) Evaluate the following integrals:

a) $\int_{-1}^{3} \frac{1}{2x+3}\,dx$ b) $\int_1^3 \frac{\cos(\ln x)}{x}\,dx$

c) $\int_0^{\pi} \frac{\cos x}{3+4\sin x}\,dx$ d) $\int_0^1 \frac{5x^3}{\sqrt{1-x^8}}\,dx$

So far we have developed the method of integration by substitution by turning the chain rule into an integration formula. Let us do the same for the product rule. We know that the derivative of the product of two functions f(x) and g(x) is given by

$$\frac{d}{dx}[\,f(x)\,g(x)\,] = g(x)f'(x) + f(x)\,g'(x)$$

where the dashes denote differentiation w.r.t.x. Corresponding to this formula we have a rule called integration by parts.

9.3.2 Integration by Parts

Let us now discuss this method of integration by parts in detail. We begin by taking two functions f(x) and g(x). Let G(x) be an anti-derivative of g(x). That is,

$$\int g(x)\,dx = G(x) \text{ or } G'(x) = g(x)$$

Then, by the product rule for differentiation we have,

$$\frac{d}{dx}[\,f(x)\,G(x)\,] = f(x)G'(x) + f'(x)\,G(x)$$
$$= f(x)g(x) + f'(x)G(x)$$

Integrating both sides we get,

$$f(x)G(x) = \int f(x)\,g(x)\,dx + \int f'(x)G(x)\,dx$$

$$\text{or } \int f(x)\,g(x)\,dx = f(x)G(x) - \int f'(x)\,G(x)\,dx$$

$$\text{Thus, } \int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int f'(x)\left\{\int g(x)\,dx\right\}dx \quad \text{..... (14)}$$

The integration done by using rule (14) is called integration by parts. In words, it can be stated as follows:

The integral of the product of two functions = first function × integral of the second function − integral of (differential coefficient of the first × integral of the second).

From (14), it can be easily seen that,

$$\int_a^b f(x)g(x)\,dx = f(x)\int_a^b g(x)\,dx - \int_a^b f'(x)\left\{\int g(x)\,dx\right\}dx$$

We now illustrate this method through some examples.

Example 11 : Integrate xe^x with respect to x.

Solution : We use integration by parts.

Step 1 : Take $f(x) = x$ and $g(x) = e^x$.

Then, $f'(x) = 1$ and $\int e^x dx = e^x$.

Step 2 : By formula (14) we have

$$\int xe^x dx = xe^x - \int 1.e^x\,dx$$

or $\int xe^x dx = xe^x - e^x + c$.

You may be wondering why we chose $f(x) = x$ and $g(x) = e^x$, and not the other way round (i.e. $f(x) = e^x$ and $g(x) = x$). Remember, our objective was to find the given integral as easily as possible. Had we chosen $f(x) = e^x$ and $g(x) = x$, and applied the method of integration by parts, we would have got

$$\int xe^x\,dx = \frac{x^2e^x}{2} - \int \frac{e^x x^2}{2}\,dx.$$

But now, the resulting integral is more difficult than the given integral. Hence, our earlier choice for f(x) was a wiser one. The next example shows how integration by parts can be used to compute a reasonably complicated integral.

Example 12 : Obtain $\int \frac{xe^{2x}}{(2x+1)^2}\,dx$.

Solution : Step 1 : Take $f(x) = xe^{2x}$ and $g(x) = \frac{1}{(2x+1)^2}$.

Then, $f'(x) = e^{2x} + 2xe^{2x} = e^{2x}(1+2x)$ and

$$\int g(x)\,dx = \frac{1}{(2x+1)^2}\,dx = -\frac{1}{2(2x+1)}$$

Now, the given integral is of the form $\int f(x)\,g(x)\,dx$.

Step 2 : We apply (14) to evaluate the given integral by parts :

$$\therefore \int \frac{xe^{2x}}{(2x+1)^2}\,dx = -\frac{xe^{2x}}{2(2x+1)} - \int e^{2x}(2x+1)\,.\left\{\frac{-1}{2(2x+1)}\right\}dx$$

$$= -\frac{xe^{2x}}{2(2x+1)} + \frac{1}{2}\int e^{2x}\,dx$$

$$= -\frac{xe^{2x}}{2(2x+1)} + \frac{1}{2}\,\frac{e^{2x}}{2} + c$$

$$= -\frac{e^{2x}}{4(2x+1)} + c.$$

You may now try this exercise :

E6) Integrate the following functions with respect to x.

a) $x \cos nx$ b) $x^2 \ln x$ c) $\frac{\ln x}{x^2}$ d) $x\,\text{cosec}^2 x$

Sometimes we need to integrate by parts more than once. We now illustrate it through the following example.

Example 13 : $\int x^2 \cos x \, dx$.

Solution : Step 1 : Take $f(x) = x^2$ and $g(x) = \cos x$

Then, $f'(x) = 2x$ and $\int \cos x \, dx = \sin x$

Therefore, the given integral is of the form $\int f(x)g(x) \, dx$

Step 2 : Integrating $\int x^2 \cos x \, dx$ by parts we get

$$\int x^2 \cos x \, dx = x^2 \int \cos x \, dx - \int 2x \left\{ \int \cos x \, dx \right\} dx$$

$$= x^2 \sin x - 2\int x \sin x \, dx + c_1 \quad \ldots\ldots\ldots \quad (15)$$

c_1 is a constant of integration.

Step 3 : Integrating $\int x \sin x \, dx$, again by parts, we get,

$$\int x \sin x \, dx = x \int \sin x \, dx - \int 1 \left\{ \int \sin x \, dx \right\} dx$$

$$= x(-\cos x) + \int \cos x \, dx + c_2$$

Therefore, $\int x \sin x \, dx = -x \cos x + \sin x + c_2 \quad \ldots\ldots\ldots \quad (16)$

c_2 being the constant of integration.

Step 4 : From (15) and (16) we get,

$$\int x^2 \cos x \, dx = x^2 \sin x - 2(-x \cos x + \sin x + c_2) + c_1$$

$$= x^2 \sin x + 2x \cos x - 2\sin x + c$$

where we have written c for $c_1 - 2c_2$.

In certain cases we apply the method of integration by parts even though the integrand is not the product of two functions. In such cases we take one of the factors as unity. The following example illustrates this point.

Example 14 : Find $\int \ln x \, dx$.

Solution : Step 1 : Note that the integrand is not a product of two functions of x. In order to apply the method of integration by parts, we take $f(x) = \ln x$, say, and

$g(x) = 1$, then $f'(x) = \frac{1}{x}$ and $\int g(x) \, dx = \int dx = x$.

Thus, the given integral is of the form $\int f(x) \, g(x) \, dx$.

Step 2 : Applying (14) we get

$$\int \ln x \, dx = \int \ln x.1 \, dx = - \int \frac{1}{x} \cdot x \, dx$$

$$= x \ln x - x + c.$$

We now consider some examples of integrals which occur quite frequently and can be integrated by parts.

Example 15 : Find $\int e^{ax} \cos bx \, dx$.

Solution : Step 1 : Choose $f(x) = e^{ax}$ and $g(x) = \cos bx$, then integration by parts gives

$$\int e^{ax}\cos bx\, dx = e^{ax}\frac{\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}\,dx + c_1 \text{ .... } (17)$$

Step 2 : Integrating $\int e^{ax}\sin bx\, dx$ by parts again we get,

$$\int e^{ax}\sin bx\, dx = e^{ax}\frac{(-\cos bx)}{b} - \int ae^{ax}\frac{(-\cos bx)}{b}\,dx$$

$$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\, dx + c_2.$$

Note that the second term on the right hand side is nothing but a constant multiple of the given integral.

Step 3 : Substituting the value of $\int e^{ax}\sin bx$ in (17) we have,

$$\int e^{ax}\cos bx\, dx = e^{ax}\frac{\sin bx}{b} - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx + c_2\right] + c_1$$

$$= e^{ax}\frac{\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\, dx + c_3 \text{ ...... } (18)$$

where $c_3 = c_1 - \frac{a}{b}c_2$.

Step 4 : Transposing the last term from the right of (18) to left we get

$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\cos bx\, dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx + c_3$$

Dividing by $\left(1 + \frac{a^2}{b^2}\right)$ we finally get

$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx) + c$, where, $c = \frac{c_3}{a^2+b^2}$ as the required integral. Similarly, the integral of the type $\int e^{ax}\sin bx\, dx$ can be obtained.

E7) Find $\int e^{ax}\sin bx\, dx$.

It may be observed that the constant of integration c would be added right at the end after all the steps have been executed.

Some examples of definite integrals which are treated by the method of integration by parts are discussed below

Example 16 : Evaluate $\int_0^{\pi/2} x \sin x\, dx$.

Solution : Step 1 : Take $f(x) = x$, so that $f'(x) = 1$.

Also let $g(x) = \sin x$ so that $\int \sin x\, dx = -\cos x$.

Therefore, the given integral is of the form

$\int_a^b f(x)\, g(x)\, dx$ with $a = 0$ and $b = \pi/2$.

Step 2 : Integration by parts gives

$$\int_0^{\pi/2} x \sin x \, dx = \Big[-x \cos x\Big]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot(-\cos x)\, dx$$

$$= 0 + \int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2}$$

$$= \sin \frac{\pi}{2} - \sin 0 = 1.$$

Thus, $\int_0^{\pi/2} x \sin x \, dx = 1.$

Example 17 : Evaluate $\int_0^5 \frac{x}{\sqrt{x+4}} dx.$

Solution : Step 1 : Let $f(x) = x$, $g(x) = \frac{1}{\sqrt{x+4}}$,

so that $f'(x) = 1$ and $\int g(x)\, dx = \int \frac{1}{\sqrt{x+4}} dx = 2(x+4)^{1/2}$

The given integral is of the form $\int_a^b f(x)g(x)\, dx$ with $a = 0$ and $b = 5$.

Step 2 : Integration by parts gives

$$\int_0^5 \frac{x}{\sqrt{x+4}} dx = 2(x+4)^{1/2} \Big|_0^5 - \int_0^5 1.2(x+4)^{1/2} dx$$

$$= 2x(x+4)^{1/2} \Big|_0^5 - \frac{4}{3}(x+4)^{1/2} \Big|_0^5$$

$$= \left[10.(9)^{1/2} - 0\right] - \left[\frac{4}{3}(9)^{3/2} - \frac{4}{3}(4)^{3/2}\right]$$

$$= 30 - \left[36 - \frac{32}{3}\right] = \frac{14}{3}\,..$$

Sometimes it may happen that both the methods of integration, namely the method of substitution and the method of integration by parts have to be applied in the same problem as illustrated here.

Example 18 : Integrate $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$ with respect to x.

Solution : Step 1 : Put $x = \cos\theta$, then $dx = -\sin\theta\, d\theta$.

Therefore, $\frac{1-x}{1+x} = \frac{1-\cos\theta}{1+\cos\theta} = \tan^2(\frac{\theta}{2})$ $\left(\text{since } \cos\theta = 1-2\sin^2(\frac{\theta}{2}) = 2\cos^2(\frac{\theta}{2}) - 1\right)$

$$\Rightarrow \sqrt{\frac{1-x}{1+x}} = \tan\theta/2$$

$$\Rightarrow \tan^{-1}\sqrt{\frac{1-x}{1+x}} = \frac{\theta}{2}$$

Step 2 : $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\, dx = \int \frac{\theta}{2}(-\sin\theta)\, d\theta$

$$= -\frac{1}{2}\int \theta \sin\theta \, d\theta \quad ..... \; (19)$$

Take $f(\theta) = \theta$ and $g(\theta) = \sin\theta$

Step 3 : Integration by parts gives

$$\int \theta \sin\theta \, d\theta = -\theta\cos\theta + \int 1.\cos\theta \, d\theta$$

$$= -\theta\cos\theta + \sin\theta \qquad \ldots\ldots (20)$$

Step 4 : From (19) and (20) we have

$$\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx = -\frac{1}{2}\left[-\theta\cos\theta + \sin\theta\right] + c$$

$$\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx = -\frac{1}{2}\left[-x\cos x^{-1}x + \sqrt{1-x^2}\right] + c$$ (because $x = \cos\theta$, and $\sin^2\theta = 1-\cos^2\theta$).

And now some exercises for you.

E8) Integrate the following functions with respect to x.

a) x^2e^x b) $e^{3x}\cos 4x$ c) $\sin^{-1}x$

d) $\tan^{-1}x$ e) $x\tan^{-1}x$ f) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

g) $\frac{e^{m}\tan^{-1}x}{(1+x^2)^{3/2}}$

E9) Evaluate the following integrals :

a) $\int_0^1 xe^{2x}\,dx$ b) $\int_0^{\pi/2} x^2\sin x\,dx$ c) $\int_{-2}^{2} 2x\sin(x^2)\,dx$

d) $\int_1^4 \ln x\,dx$ e) $\int_1^3 x^2 e^{x^3}\,dx$

We have given you the basic formulas for integrals involving trigonometric functions (Table 1). You have also become familiar with the two methods of integration, namely, integration by parts and integration by the method of substitution. We shall now obtain the integrals of some more trigonometric functions using these methods and basic formulas.

9.4 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Many trigonometric integrals can be evaluated after transformations of the integrand into the most familiar trigonometric formulas. In this section, we shall consider some examples of this type.

Integrals of the form $\int \sin^m x \cos^n x\,dx$, where integer m and n can be positive, negative or zero. For different values of m and n different substitutions are chosen as given below.

i) if m is an odd positive integer, we substitute $\cos x = t$,

ii) if n is an odd positive integer, we substitute $\sin x = t$.

iii) if $m+n$ is an even negative integer, then the substitution $\tan x = t$, reduces the integrand into a sum of powers of t.

For better understanding of these cases, let us take a few examples.

Example 19 : Obtain $\int \sin^5 dx$.

Solution : Here m = 5 and n = 0.

Put $\cos x = t$. Then $-\sin x\, dx = dt$

Therefore, $\int \sin^5 x\, dx = -\int \sin^4 x \,.\, (-\sin x\, dx)$

$$= -\int (1-\cos^2 x)^2 \,.\, (-\sin x\, dx)$$

$$= -\int (1-t^2)^2\, dt$$

$$= -\int (t^4 - 2t^2 + 1)\, dt$$

$$= - \frac{t^5}{5} + \frac{2t^3}{3} - t + c$$

$$= -\left[\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x + c\right]$$

Example 20 : Find $\int \sin^4 x \cos^3 x\, dx$.

Solution : Here m= 4 and n = 3. Since n is an odd integer, we put $\sin x = t$ and $\cos x\, dx = dt$.

Therefore, $\int \sin^4 x \cos^3 x\, dx = \int \sin^4 x \cos^2 x\, (\cos x\, dx)$

$$= \int t^4\, (1-t^2)\, dt$$

$$= \int (t^4 - t^6)\, dt = \frac{t^5}{5} - \frac{t^7}{7} + c$$

$$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c.$$

Thus, $\int \sin^4 x \cos^3 x\, dx = \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

Let us look at another example.

Example 21 : Obtain $\int \frac{1}{\sin^4 x \cos^2 x}\, dx$.

Solution : Here m = −4, n = −2 and m+n = −4−2 = −6, an even negative integer.

Put $\tan x = t$. Then $\sec^2 x\, dx = dt$.

Therefore, $\int \frac{1}{\sin^4 x \cos^2 x}\, dx = \int \frac{1}{\frac{\sin^4 x}{\cos^4 x}\cos^4 x} \,.\, \sec^2 x\, dx$

(Multiply and divide by $\cos^4 x$)

$$= \int \frac{1}{\tan^4 x} \sec^4 x \sec^2 x\, dx,$$

$$= \int \frac{(1+\tan^2 x)^2}{\tan^4 x} \sec^2 x\, dx, \text{ (because } 1+\tan^2 x = \sec^2 x)$$

$$= \int \frac{(1+t^2)^2}{t^4}\, dt$$

$$= \int \frac{1+t^4+2t^2}{t^4}\, dt$$

$$= \int (t^{-4}+1+2t^{-2})\, dt$$

$$= -\frac{t^{-3}}{3} + t - 2t^{-1} + c$$

$$= t - \frac{1}{3t^3} - \frac{2}{t} + c$$

$$= \tan x - \frac{1}{3\tan^3 x} - \frac{2}{\tan x} + c$$

$$\int \frac{1}{\sin^4 x \cos^2 x}\, dx = \tan x - \frac{1}{3}\cot^3 x - 2\cot x + c$$

Now you can do this exercise easily.

E10) Integrate the following functions with respect to x.

a) $\cos^2 x \sin^3 x$

b) $(\frac{1}{\sin x}) \cos^3 x$

c) $\sin x (\cos x)^{-5}$

At this stage you may ask, what happens if m and n are not of the type mentioned in (i) – (iii) above ? To answer this we consider the general integral $\int \sin^m x \cos^n x\, dx$ and evaluate it

we write,

$\sin^m x \cos^n x = \sin^m x \cos x \cos^{n-1} x$

Then, $\int \sin^m x \cos^n x\, dx = \int \sin^m x \cos x \cos^{n-1} x\, dx.$

Integrating the right hand side by parts by taking $\cos^{n-1} x$ as the first function and $\sin^m x \cos x$ as the second, we get

$$\int \sin^m x \cos x \cos^{n-1} x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{m+1} x \cos^{n-2} x \sin x\, dx$$

$$(\text{since } \int \sin^m x \cos x\, dx = \frac{\sin^{m+1} x}{m+1})$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x (1-\cos^2 x)\, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\, dx - \frac{n-1}{m+1}\int \sin^m x \cos^n x\, dx.$$

Transposing the last term on the right hand side and dividing by $1 + \frac{n-1}{m+1} = \frac{m+n}{m+1}$ we get

$$\int \sin^m x \cos^n x\, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\, dx \qquad \text{......... (20)}$$

as the required integral formula. This is called the **reduction formula** for $\int \sin^m x \cos^n x\, dx$ and is usually denoted by $I_{m,n}$. By writing $\int \sin^m x \cos^n x\, dx$ as $\int \sin^{m-1} x \cos^n x \sin x\, dx$ and proceeding as above, we can obtain another form of $I_{m,n}$ as.

$$\int \sin^m x \cos^n x\, dx = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx \qquad \text{......... (21)}$$

In particular putting m = 0 in (20) and n = 0 in (21) we obtain the following reduction formulas.

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx \qquad \text{......... (22)}$$

and $\int \sin^m x\, dx = \frac{-\sin^{m-1}x \sin \dot{x}}{m} + \frac{m-1}{m} \int \sin^{m-2}x\, dx$ .... (23)

Example 22 : Obtain $\int \sin^4 x \cos^2 x\, dx$.

Solution : Here m = 4 and n = 2 in (20).

$$\therefore \int \sin^4 x \cos^2 x\, dx = \frac{1}{6}\sin^5 x \cos x + \frac{1}{6}\int \sin^4 x\, dx \text{ by (20)}$$

$$= \frac{1}{6}\sin^5 x \cos x + \frac{1}{6}\left[-\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\int \sin^2 x\, dx\right] \text{ by (23)}$$

$$= \frac{1}{6}\sin^5 x \cos x + \frac{1}{24}\sin^3 x \cos x + \frac{1}{8}\left(-\frac{1}{2}\sin x \cos x + \frac{1}{2}\int dx\right) \text{ by (23)}$$

Hence, $\int \sin^4 x \cos^2 x\, dx = \frac{1}{48}(8\sin^5 x - 2\sin^3 x - 3\sin x)\cos x + \frac{1}{16}x$.

Also, the definite integral $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$ can be evaluated using formula (20) in the form,

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{n-1}{m+1}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx \quad \text{......... (24)}$$

The first term on the right of (20) is zero, because sin 0 = 0, cos $\pi/2$ = 0. Similarly, from (22) and (23) we get, respectively,

$$\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\, dx \quad \text{......... (25)}$$

and $\int_0^{\pi/2} \sin^m x\, dx = \frac{m-1}{n}\int_0^{\pi/2} \sin^{m-2} x\, dx$ .... (26)

Definite integral (24) gives the general formula for all integral values of m and n. Particular forms of this can be obtained by giving different values to m and n. We now consider some of these particular cases.

(I) Let m and n be even positive integers. In this case applying formula (24) repeatedly till the power of cos x becomes zero and afterwards applying formula (26) repeatedly, we get,

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{n-1}{m+1}\cdot\frac{n-3}{m+n-2}\cdot\frac{n-5}{m+n-4}\cdots\frac{1}{m+2}\int_0^{\pi/2}\sin^m x\, dx$$

$$= \frac{(n-1).(n-3).(n-5)\;.....}{(m+n)(m+n-2)(m+n-4)}\,\frac{1}{(m+2)}\cdot\frac{m-1}{m}\cdot\frac{m-3}{m-2}\cdot\frac{1}{2}\int_0^{\pi/2} dx$$

$$= \frac{(m-1)(m-3)(m-5)\;.1.(n-1)(n-3)\;\cdots}{(m+n)(m+n-2)(m+n-4).......}\,\frac{1}{2}\cdot\frac{\pi}{2} \quad \text{......... (27)}$$

(II) Let n be even and m an odd positive integer.

Proceeding as in case I we get

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)....2.(n-1)(n-3)\;....1}{(m+n)(m+n-2)(m+n-4).......\;3.1}\cdot 1 \quad \text{......... (28)}$$

From I and II above, the formulas when either of m or n is zero can be written down easily.

Other particular cases of (24) can be dealt with in the same manner.

Let us now do a few examples using these formulas.

Example 23 : Evaluate $\int_0^{\pi/2} \sin^7 x \, dx$.

Solution : Using (28) with m = 7 and n = 0 we get

$$\int_0^{\pi/2} \sin^7 x \, dx = \frac{(7-1)}{7} \cdot \frac{(7-3)}{(7-2)} \cdot \frac{(7-5)}{(7-4)} = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{16}{35}.$$

Example 24 : Evaluate $\int_0^{\pi/2} \sin^8 x \, dx$.

Solution : Using (27) with m = 8 and n = 0 we get

$$\int_0^{\pi/2} \sin^8 x \, dx = \frac{(8-1)}{8} \cdot \frac{(8-3)}{(8-2)} \cdot \frac{(8-5)}{(8-4)} \cdot \frac{(8-7)}{(8-6)} \cdot \frac{\pi}{2}$$
$$= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35\pi}{256}$$

Note that only the integrals involving even powers of sine or cosine when integrated from 0 to $\pi/2$ are multiplied at the end by a factor of $\pi/2$.

Let us look at another example.

Example 25 : Evaluate $\int_0^{\pi/2} \sin^6 x \cos^8 x \, dx$.

Solution : Here m = 6 and n = 8 both are even, so formula (27) gives

$$\int_0^{\pi/2} \sin^6 x \cos^8 x \, dx = \frac{(6-1)}{(6+8)} \cdot \frac{(6-3)}{(6+8-2)} \cdot \frac{(6-5)}{(6+8-4)} \cdot \frac{(8-1)}{(6+8-6)}$$
$$\frac{(8-3)}{(6+8-8)} \cdot \frac{(8-5)}{(6+8-10)} \cdot \frac{(8-7)}{(6+8-12)} \cdot \frac{\pi}{2}$$
$$= \frac{5}{14} \cdot \frac{3}{12} \cdot \frac{1}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{4096}$$

You may now try the following exercise.

E11) Evaluate the following integrals:

a) $\int_0^{\pi/4} \sin^4 2x \, dx$ b) $\int_0^{\pi/2} \cos^8 x \, dx$ c) $\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx$

d) $\int_0^{\pi/2} \sin^6 x \, dx$ e) $\int_0^{\pi/2} \cos^5 x \sin^4 x \, dx$

We now end this unit by giving a summary of what we have done in it.

9.5 SUMMARY

In this unit we have covered the following points.

1) Table of standard integrals.
2) Method of integration by substitution.
3) Integration with respect to x, of functions of the form $ax+b$, x^n, $\{f(x)\}^n f'(x)$, $\frac{f'(x)}{f(x)}$, $a^2 \pm x^2$.
4) Method of integration by parts and its applications.
5) Evaluation of definite integrals.
6) Integration of $\int \sin^m x \cos^n x \, dx$ when m and n are integers (even or odd).

7) Evaluation of $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$ when either m even, n odd or m odd and n even or both even or both odd.

9.6 SOLUTIONS/ANSWERS

E1) a) $\sec x + c$

b) $-\cot x - x + c$

c) $-\frac{a}{x} + b \ln x + c$

d) $\frac{2a}{7}x^{7/2} + \frac{2b}{5}x^{5/2} + \frac{2d}{3}x^{3/2} + c$

e) $\frac{x^3}{3} - x + 2\tan^{-1}x + c$

f) $\frac{x^{a+1}}{a+1} + \frac{a^x}{\ln a}$

E2) a) $\ln 2$

b) $\frac{\pi}{2} - 1$

c) $\frac{\pi}{2}$

E3) a) $\frac{(3x-2)^7}{12} + c$

b) $-\frac{1}{3}(3-2x)^{3/2} + c$

c) $\frac{1}{2}e^{2x+3} + c$ (Hint : Use the substitution $2x+3 = t$)

d) $-(\frac{1}{m}) \cos mx + c$

E4) a) $\frac{1}{2}\tan x^2 + c$

b) Put $\sin^{-1}x = t$ so that $\frac{1}{\sqrt{1-x^2}}dx = dt$ and

$$\int \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx = \int t^2 \, dt = \frac{t^3}{3} + c = \frac{1}{3}(\sin^{-1}x)^3 + c$$

c) $\frac{1}{4}(1+\ln x)^4 + c$

d) $-\ln(1+\cot x) + c$

e) $\left(\frac{1}{2\sqrt{2}}\right)\sin^{-1}\left(\sqrt{2}\,x^2\right) + c$

E5) a) $\ln 3$

b) Put $\ln x = t$ and $\frac{1}{x}dx = dt$, so that

$$\int_0^3 \frac{\cos(\ln x)}{x}dx = \int_0^{\ln 3} \cos t \, dt = \sin t \Big|_0^{\ln 3} = \sin(\ln 3)$$

c) $\frac{1}{4}\ln(1 + \frac{2}{\sqrt{3}})$

d) $\frac{5\pi}{8}$

E6) a) $\frac{(nx \sin nx + \cos nx)}{n^2} + c$

b) $(3\ln x - 1) \cdot \frac{x^3}{9} + c$

c) $\frac{-(\ln x + 1)}{x} + c$

d) $\ln \sin x - x \cot x + c$

E7) $\frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

E8) a) $e^x(x^2-2x+2) + c$

b) $\frac{e^{3x}(4\sin 4x + 3\cos 4x)}{25} + c$

c) $x\sin^{-1}x + \sqrt{(1-x^2)} + c$

d) $x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$

e) $\int x \tan^{-1}x \, dx$

$= \tan^{-1}x \frac{x^2}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2} dx$ (using integration by parts with $f(x) = \tan^{-1}x$, $g(x) = x$)

$= \tan^{-1}x \cdot \frac{x^2}{2} - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$

$= \frac{x^2}{2}\tan^{-1} - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + c$

$= \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x + c$

f) $2x \tan^{-1}x - \ln(1+x^2) + c$

g) $\frac{e^{m\phi}}{\sqrt{m^2+1}}\cos(\phi - \cot^{-1}m) + c$, where $\phi = \tan^{-1}x$.

E9) a) $\frac{1}{4}(e^2+1)$ b) $(\pi-2)$ c) 0

d) $4\ln 4 - 3$ e) $\frac{1}{3}(e^{27}-e)$

E10) a) $\frac{1}{5}\cos^5x - \frac{1}{3}\cos^3x + c$

b) $\ln \tan x + \frac{1}{2}\tan^2x + c$

c) Here $m = 3$ and $n = -5$ in the integral $\int \sin^m x \cos^n x \, dx$ so that $(m+n) = -2 \doteq$ even negative integer, now

$\int \sin^3x(\cos x)^{-5} dx = \int \frac{\sin^3x}{\cos^3x} \cdot \frac{1}{\cos^2x} dx = \int \tan^3x \sec^2x \, dx$

put $\tan x = t$ and $\sec^2x \, dx = dt$, then

$\int \tan^3x \sec^2x \, dx = \int t^3 dt = \frac{t^4}{4} + c = \frac{(\tan^4 x)}{4} + c$

E11) a) Put $2x = t$ so that $2dx = dt$

and $\int_0^{\pi/4} \sin^4 2x \, dx = \frac{1}{2}\int_0^{\pi/4} \sin^4 t \, dt = \frac{3\pi}{32}$

b) $\frac{35\pi}{256}$ c) $\frac{2}{35}$ d) $\frac{5\pi}{32}$ e) $\frac{8}{315}$

UNIT 10 DIFFERENTIAL EQUATIONS

Structure

10.1 INTRODUCTION

Analysis has been dominant branch of mathematics for 300 years, and differential equations is the heart of analysis. Differential equation is an important part of mathematics for understanding the physical sciences. It is the source of most of the ideas and theories which constitute higher analysis. Many interesting geometrical and physical problems are proposed as problems in differential equations, and solutions of these equations give complete picture of the state of these problems.

Differential equations work as a powerful tool for solving many practical problems of science as well as a wide range of purely mathematical problems. In Units 6 and 7 we defined first and higher order ordinary and partial derivatives of a given function. In this unit, we make use of these derivatives and first introduce some basic definitions. Then we discuss the formation of differential equations. We also discuss various techniques of solving some important types of differential equations of the first order and first degree. We have discussed the method of separation of variables together with methods of solving homogeneous, exact and linear differential equations. Also, differential equations which are reducible to homogeneous for or equations reducible to linear form are considered.

Objectives

After studying this unit you should be able to

- distinguish between ordinary and partial differential equations and between the order and the degree of an equation,
- form a differential equation whose solution is given,
- use the method of separation of variables,
- identify homogeneous or linear equations and solve the same,
- verify whether a given differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact or not, and solve exact equations,
- identify an integrating factor in some simple cases which makes the given equation exact.

10.2 PRELIMINARIES

In this section we shall define and explain the basic concepts in differential equations and illustrate them through examples. Recall that given an equation or relation of the type $f(x, y) = 0$, involving two variables x and y, where $y = y(x)$, we call x the **independent variable** and y the **dependent variable** (ref. Unit 1 of Block 1). Any equation which gives the relation between the independent variable and the derivative of the dependent variable with respect to the independent variable, is a differential equation. In general we have the following definition.

Definition : A differential equation is an equation that involves derivatives of dependent variable with respect to one or more independent variables.