# UNIT 7 APPLICATIONS OF DIFFERENTIAL CALCULUS

## Structure



## 7.1 INTRODUCTION

In Unit 6, we obtained derivatives of various types of functions and also derived rules for obtaining these derivatives. In this unit the main emphasis will be on the applications of derivatives.

In all branches of science we often face problems like : (i) How can we find accurate values of a function f corresponding to given values of x? (ii) How can we find the maximum and minimum values of a function f in a certain domain? A simple way of tackling such problems is through the application of differentiation. Consider, for instance, the first question above. One of the ways of calculating functional values to a certain degree of accuracy is through the expansion of the function as in a power series. The method of Maclaurin expansion is one such technique, which has been explained in this unit. We shall also explain the process of finding second derivatives, the maxima and minima of a function and of tracing a given curve. We shall also discuss functions of two variables.

To start with, we have talked about the problem of finding tangents and normals to a given curve, which are geometrical applications of differentiation.

### Objectives

After reading this unit, you should be able to

- write the equation of the tangent and the normal to a given curve at a given point,
- compute the second and higher order derivatives of a given function,
- write the power series expansion of some functions,
- compute the maxima and minima of various functions,
- identify and draw the graphs of some significant curves,
- find the first and second order partial derivatives of a function of two variables given in explicit, implicit or parametric form.

## 7.2 TANGENTS AND NORMALS

You have already studied in Unit 6, that if a curve is given by the equation $y = f(x)$ where $f(x)$ has a derivative $f'(x)$ at every point in the domain of f, then the derivative at a point P of the curve is the slope of the tangent to the curve at the point P (see Fig. 1). Thus, it is now simple for us to find the equation of the tangent at any point of the curve by using the point slope form.

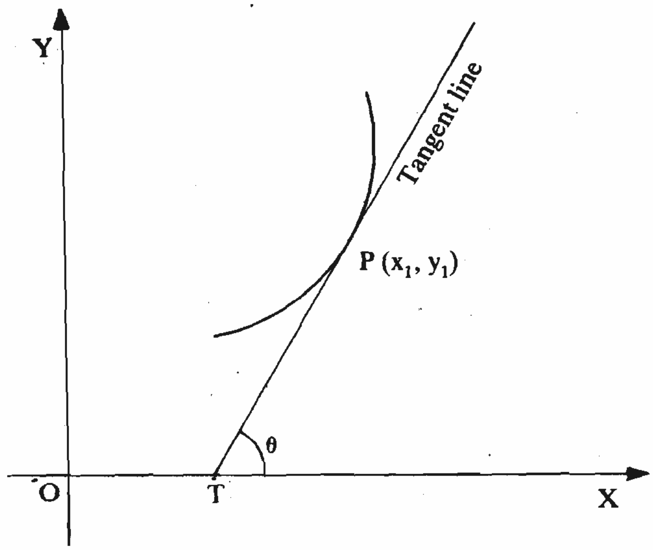


Fig. 1

### 7.2.1 Tangent to a Curve at a Point

You already know from your study of coordinate geometry (Unit 4 of Block 1) that the equation to any line through a point $P(x_1,y_1)$ is given by

$y - y_1 = m(x - x_1)$, where m is the slope of the line. But, the slope m of the tangent at $P(x_1,y_1)$ to the curve $y = f(x)$ is the value of $\frac{dy}{dx}$ at $(P(x_1,y_1)$. This slope can also be written as $\left.\frac{dy}{dx}\right|_P$ or $\left(\frac{dy}{dx}\right)_P$. Therefore, you can at once write down the **equation to the tangent** PT in the form

$$y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$$

as illustrated in the following example.

**Example 1:** Find the equation of the tangent to the circle $x^2 + y^2 = 25$, at the point (3,4).

**Solution :** It is clear that the point (3,4) lies on the given circle. The equation to the tangent at (3,4) will be

$$(y - 4) = \left(\frac{dy}{dx}\right)_{(3,4)} (x-3)$$

Differentiating $x^2 + y^2 = 25$ with respect to x, we get

$$2x + 2y\frac{dy}{dx} = 0, \text{ i.e., } \frac{dy}{dx} = -\frac{x}{y}$$

thus, $\left(\frac{dy}{dx}\right)_{(3,4)} = \frac{-3}{4}$.

Therefore, the equation to the tangent is $(y-4) = \frac{-3}{4}(x-3)$, which on simplification becomes $3x + 4y = 25$.

In a similar way, we now define the normal to a curve at a point.

### 7.2.2 Normal to a Curve at a Point

We again recall from Unit 4 that the slope of the normal line at a given point is the negative of the reciprocal of the slope of tangent line at that point.

The **normal** at a point P to a curve is defined as the straight line through P which is perpendicular to the tangent to the curve at P. If PT is the tangent at P as in Fig. (2), then PN which is perpendicular to PT through P is the normal at P.

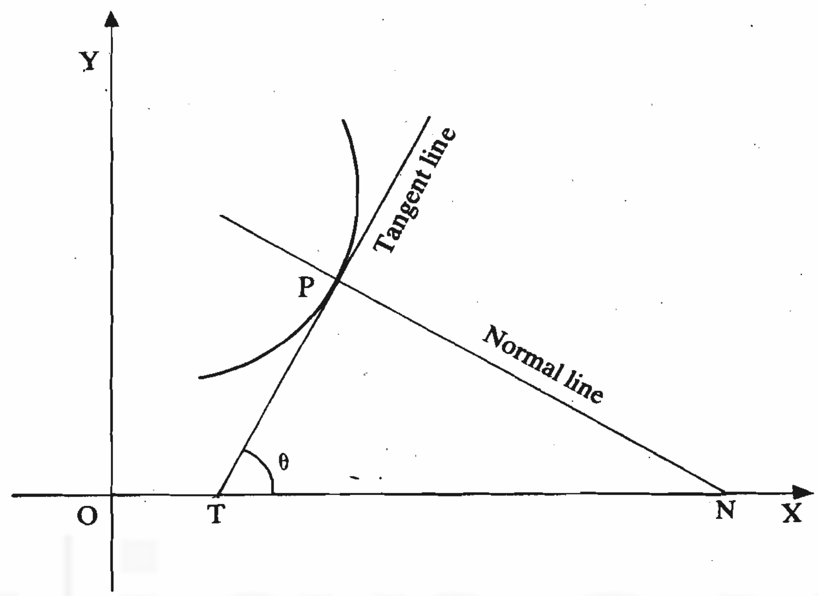


Fig. 2

You know from coordinate geometry (Unit 4) that two straight lines having slopes m and $m_1$, respectively are perpendicular to each other iff $mm_1 = -1$. Therefore, if the slope of the tangent PT is m, then the slope of the normal, PN, is $-\frac{1}{m}$. In other words, the slope of a normal to a curve at a point is given by $\frac{-1}{dy/dx}$, where $\frac{dy}{dx}$ represents the slope of the tangent to the curve at the same point.

Thus, you can write down the **equation of the normal** PN as :

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_P}(x - x_1).$$

For example, the equation of the normal at (3,4) to the circle $x^2 + y^2 = 25$ of Example 1 is,

$$y - 4 = -\frac{1}{\left(\frac{-3}{4}\right)}(x - 3) = \frac{4}{3}(x - 3)$$

which is $3y - 4x = 0$.

You can now try the following exercises.

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E1) Find the equations of the tangent and the normal to the curve $y = 4x - x^2$ at (1,3).

E2) Find the equations of the tangent and the normal to the following curves at $P(x_1,y_1)$.

a) $x^2 + y^2 = a^2$.

b) $xy = k^2$.

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In Unit 6 you have already learnt to differentiate a function when its equation is given in parametric form. Let us now obtain the equations of tangents and normals for such functions.

### 7.2.3 Tangents and Normals in Parametric Form

Suppose the equation of the curve is given in terms of the parameter t in the form
$x = \phi(t)$ ; $y = \psi(t)$

then, the slope $\left.\frac{dy}{dx}\right|_P = \left.\frac{(dy/dt)}{(dx/dt)}\right|_P = \left.\frac{\psi'(t)}{\phi'(t)}\right|_P$ (Using chain rule)

Therefore, the equation of the tangent at P ($t = t_1$) is

$$y - \psi(t_1) = \frac{\psi'(t_1)}{\phi'(t_1)} \{ x - \phi(t_1) \}$$

Similarly, the equation of the normal at P ($t_1$) is of the form

$$y - \psi(t_1) = -\frac{\phi'(t_1)}{\psi'(t_1)} \{ x - \phi(t_1) \}.$$

as illustrated by the following example.

**Example 2 :** Find the equations of the tangent and the normal at $\theta = \frac{\pi}{3}$ to the curve given by $x = a\sin\theta$, $y = a\cos\theta$.

**Solution :** Here, $\frac{dx}{d\theta} = a\cos\theta$ $\frac{dy}{d\theta} = -a\sin\theta$

Therefore, $\frac{dy}{dx} = -\frac{a\sin\theta}{a\cos\theta} = -\tan\theta$

At $\theta = \frac{\pi}{3}$, $\tan\theta = \sqrt{3}$.

Also, $x = a\sin\frac{\pi}{3} = \frac{a}{2}.\sqrt{3}$ and $y = a\cos\frac{\pi}{3} = \frac{a}{2}$.

$\therefore$, the equation of the tangent at $\theta = \frac{\pi}{3}$, is

$$y - \frac{a}{2} = -\sqrt{3}\left[x - \frac{a\sqrt{3}}{2}\right]$$

and the equation of the normal at $\theta = \frac{\pi}{3}$, is

$$y - \frac{a}{2} = \frac{1}{\sqrt{3}}\left[x - \frac{a\sqrt{3}}{2}\right].$$

On the similar lines you can try these exercises.

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E3) Find the equation of the tangent and the normal at $t = 2$, to the curve $x = at^2$, $y = 2at$

E4) Find the equation of the tangent and the normal at $\theta = \frac{\pi}{4}$, to the curve $x = a\sin\theta$, $y = b\cos\theta$.

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Until now we have been talking about $\frac{dy}{dx}$, that is, the first order derivative of y with respect to x. In the next section we shall talk about the higher order derivatives of y.

## 7.3 HIGHER ORDER DERIVATIVES

If $y = f(x)$ is a differentiable function of x, its derivative $\frac{dy}{dx}$ is itself a function of x. If $\frac{dy}{dx}$ is again differentiable, we denote its derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ by $\frac{d^2y}{dx^2}$, pronounced dee two y by dee x two. This is called the second order derivative or simply second derivative of y with respect to x. It is also denoted by $f''(x)$, $f^{(2)}(x)$, $y_2$ or $y''$. Similarly, if $\frac{d^2y}{dx^2}$ is differentiable we differentiate again and get $\frac{d^3y}{dx^3} = f^{(3)}(x)$. If we can continue to differentiate the function, we successively get, $f^{(4)}(x)$........$f^{(n)}$, and so on. These are also denoted by $y_3$, $y_4$,....... $y_n$ etc., and are called respectively, the third derivative, fourth derivative and nth derivative of $y = f(x)$. For obvious reasons. we call this successive differentiation. In particular the derivative of $y = f(x)$ of order n, evaluated at $x = a$, is denoted by $y_n|_{x=a}$ or $f^{(n)}(a)$ or $y_n(a)$. Consider the following examples now.

**Example 3 :** If $y = 3x^5 - 4x^3 + 2x^2 - 8$, find the derivatives of y upto 3rd order.

**Solution :** $y_1 = \frac{dy}{dx} = 15x^4 - 12x^2 + 4x$

$$y_2 = \frac{dy_1}{dx} = 60x^3 - 24x + 4$$

$$y_3 = \frac{dy_2}{dx} = 180x^2 - 24.$$

**Example 4 :** If $x = at^2$, $y = 2at$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

**Solution :** We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Here $\frac{dy}{dt} = 2a$, $\frac{dx}{dt} = 2at$. Therefore, $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{dt}{dx}$ (chain rule)

$$= \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{1}{dx/dt}$$

$$= \frac{-1}{t^2} \cdot \frac{1}{2at} = \frac{-1}{2at^3}.$$

You may now try this exercise.

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E5) Find the derivatives upto third order for the following functions:

a) $f(x) = \sqrt{x + 1}$

b) $y^2 = 4ax$

c) $y = \sin x$

d) $y = x^2 \ln x$

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Let us now find the $n^{th}$ order derivative, assuming it exists in some simple standard cases.

(1) $y = x^n$ : In this case

$y_1 = n\,x^{n-1}$

$y_2 = n(n-1)x^{n-2}$

$y_3 = n(n-1)(n-2)x^{n-3}$ and so on.

Here, we observe that $y_1$, $y_2$, $y_3$ follow a particular pattern and therefore, we can guess the $n^{th}$ order derivative $y_n$ in the form

$y_n = n(n-1)(n-2) \ldots\ldots\ldots\ldots (n-\overline{n-1})x^{n-n}$, where $\overline{n-1}$ denotes $(n-1)$

$= n(n-1)(n-2) \ldots\ldots\ldots\ldots 1$

In case, n is a positive integer we have a formula

$\frac{d^n}{dx^n}(x^n) = n!$ that is, the $n^{th}$ derivative of $x^n$ is a constant, namely, $n!$ and the $(n+1)^{th}$ derivative along with the other higher derivatives are all zero.

In case $n = -1$, we get

$y_n = (-1)(-2) \ldots\ldots (-n)x^{-1-n}$

$$\therefore \frac{d^n}{dx^n}\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$$

In particular if we have $y = \ln x$. Then we know $y_1 = \frac{1}{x}$, and

making use of the above formula we get, $y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$.

(2) $y = e^{ax}$, where a is a constant :

Here, $y_1 = ae^{ax}$, $y_2 = a^2e^{ax}$, $y_3 = a^3e^{ax}$, ..........

From this pattern, we can guess that,

$y_n = a^ne^{ax}$. Thus, $\frac{d^n}{dx^n}(e^{ax}) = a^ne^{ax}$.

In particular, if $a = 1$, then $\frac{d^n}{dx^n}(e^x) = e^x$. **This shows that the derivative of $e^x$ of any order is again $e^x$.**

(3) $y = \sin(ax+b)$, where a and b are arbitrary constants.

Now, $y_1 = a\cos(ax+b) = a\sin(ax+b+\frac{\pi}{2})$ [ because $\sin(\frac{\pi}{2}+\theta) = \cos\theta$ ]

$y_2 = a^2\cos(ax+b+\frac{\pi}{2}) = a^2\sin(ax+b+\frac{\pi}{2}+\frac{\pi}{2}) = a^2\sin(ax+b+\frac{2\pi}{2})$,

$y_3 = a^3\sin(ax+b+\frac{3\pi}{2})$, $y_4 = a^4\sin(ax+b+\frac{4\pi}{2})$, and so on.

Similarly, $y_n = a^n\sin(ax+b+\frac{n\pi}{2})$.

Thus, $\frac{d^n}{dx^n}\sin(ax+b) = a^n\sin(ax+b+\frac{n\pi}{2})$

If we put $a = 1$, $b = 0$, we get,

$\frac{d^n}{dx^n}(\sin x) = \sin(x+\frac{n\pi}{2})$.

Proceeding, similarly as in (3) above, we can show

(4) $\frac{d^n}{dx^n}\cos(ax+b) = a^n\cos(ax+b+\frac{n\pi}{2})$. In particular for $a = 1$ and $b = 0$, we have $\frac{d^n}{dx^n}(\cos x) = \cos(x+\frac{n\pi}{2})$.

(5) Let $y = a^x$. Then, from Unit 6 you know that y can be written as $y = e^{x\ln a}$. Thus, $y_1 = \ln a\, e^{x\ln a}$, $y_2 = (\ln a)^2 \cdot e^{x\ln a}$,.........

From the pattern that follow, we may write

$y_n = (\ln a)^n \cdot e^{x\ln a} = (\ln a)^n \cdot a^x$.

Having obtained the $n^{th}$ derivative of some of the standard functions, you can very easily try this exercise.

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E6) Find the $n^{th}$ order derivatives of the following functions:

a) $\frac{1}{x+a}$

b) $\ln(ax+b)$

c) $\cos\frac{x}{a}$

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Many times you might have come across expansions of various algebraic, trigonometric and exponential functions given by $y = f(x)$, in ascending integral powers of the variable x. For example,

(1) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + .........$

(Binomial theorem)

(2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \ldots\ldots\ldots\ldots$

(3) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots$

These expansions are all special cases of a general theorem called Taylor's theorem. We shall not be going into the details of this theorem as it is beyond the scope of this course. This theorem gives the expansion of f(x) in an ascending integral powers of (x−h) for a given h in the form

$$f(x) = f(h) + f'(h)(x-h) + \frac{f''(h)}{2!}(x-h)^2 + \ldots\ldots + \frac{f^n(h)}{n!}(x-h)^n + \ldots\ldots$$

A particular form of this expansion, when h = 0 is,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f''(0) + \ldots\ldots + \frac{x^n f^{(n)}}{n!}(0) + \ldots\ldots\ldots$$

which is known as **Maclaurin's expansion.** It is named after Colin Maclaurin (1696−1740), professor of mathematics at the University of Edinburgh. The expansion fails if f(x), or one of its derivatives, becomes infinity or becomes discontinuous in the domain of x. For instance, if $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$. (As $x \to 0$, $\frac{1}{2\sqrt{x}} \to \infty$). Therefore, f′(x) does not exists at x = 0, and the expansion breaks down at the second term itself. Thus, for the applicability of Maclaurin's expansion, its validity is essential.

Let us once again consider the function taken in (2) above.

**Example 5 :** Assuming the validity, find the Maclaurin's expansion for sin x.

**Solution :** We have

$f(x) = \sin x$, and $f(0) = 0$
$f^{(1)}(x) = \cos x$, $f^{(1)}(0) = 1$
$f^{(2)}(x) = -\sin x$, $f^{(2)}(0) = 0$,
$f^{(3)}(x) = -\cos x$, $f^{(3)}(0) = -1$,
$f^{(4)}(x) = \sin x$, $f^{(4)}(0) = 0$,
$f^{(5)}(x) = \cos x$, $f^{(5)}(0) = 1$,

and so on.

Hence, we have

$$\sin x = f(0) + x f^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \frac{x^3}{3!}f^{(3)}(0) + \ldots\ldots\ldots$$

or,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots$$

as the Maclaurin's expansion of f(x).

You may now try the following exercises.

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E7) Establish the following expansions assuming that expansion is valid for the function f given by

a) $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{4!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots$ for all x.

b) $f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots + \frac{x^n}{n!}\cos\frac{n\pi}{2} + \ldots\ldots$ for all x.

E8) a) Expand $\tan^{-1}x$ using Maclaurin's expansion, assuming the validity of the expansion upto $3^{rd}$ degree in x.

b) Given $\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots\ldots\ldots$

Obtain the expansion for $\frac{1}{(1+x)^3}$ by repeated differentiation of the given function.

In many applications it is necessary to find the largest or smallest values of some particular function. That is, for a given function f with domain D, find those points in D where f assumes its greatest or least values, and then evaluate f at these points. For instance, it may be required to determine a route between two points that can be travelled in minimum time. Such problems are called optimisation problems or problems of finding maxima and minima of a given function in a given domain. Let us now study these problems.

## 7.4 MAXIMA AND MINIMA

We shall first approach the concept to be developed from a geometric point of view. We shall then formulate analytic definitions and from these obtain results analytically.

### 7.4.1 Increasing and Decreasing Functions

You know that we can look at the derivative of a function in two different ways:

i) as the slope of a curve at a point on its graph and the other,

ii) as the instantaneous rate of change.

Let us see if we can conclude anything about the behaviour of the curve $y = f(x)$ at a point on it from these two aspects of the derivative.

Consider a graph with equation $y = f(x)$ as in Fig. 3, at every point of which there is a unique tangent with a finite slope.

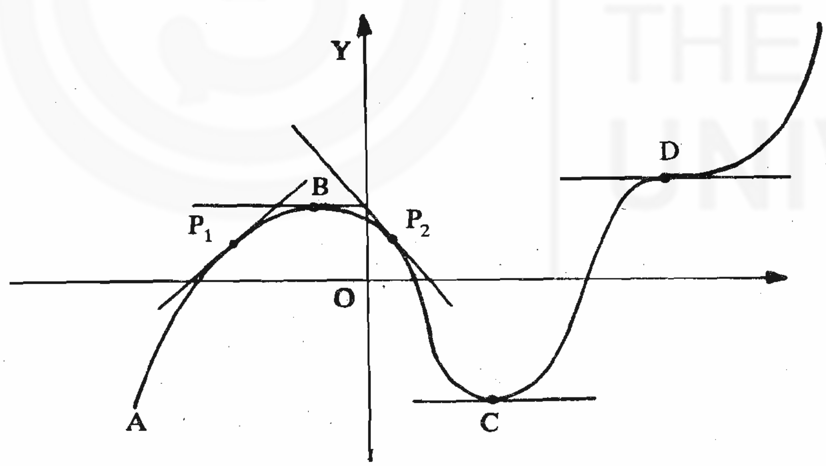


**Fig. 3**

Now if we start at point A and proceed from left to right along the graph, then the graph rises from A to B, falls from B to C, rises from C to D, and continues to rise. If we draw a tangent at any point $P_1$ between A and B (where the graph is rising) this tangent makes an acute angle with the x-axis and hence its slope is a positive number. Hence, $\frac{dy}{dx}$ (being this slope) at that point is positive. And it is clear that **at any point on the graph for which $\frac{dy}{dx}$ is positive, the graph, is rising.**

On the other hand, at $P_2$ between B and C (where the graph is falling), the tangent makes an obtuse angle with the x-axis and hence the slope will be a negative number and hence $\frac{dy}{dx}$ will be negative at that point. So that **at any point on the graph for which $\frac{dy}{dx}$ is negative, the graph is falling.**

Now at the point B, where the graph is neither rising nor falling, the tangent is parallel to the x-axis; consequently its slope is zero and hence $\frac{dy}{dx} = 0$ at that point. Such is also the case at points C and D. The three points B, C, D then have the common geometric property. We say that the graph is stationary at such points and **these points are called stationary points.**

To summarise we can say that, given a function $y = f(x)$ defined in some domain,

i) if $\frac{dy}{dx} > 0$ in any interval I of the domain, then $f(x)$ is an increasing function of x in I (i.e. y increases along with x).

ii) if $\frac{dy}{dx} < 0$ in the interval I, then $f(x)$ is a decreasing function of x in I (i.e. y decreases when x increases).

iii) if $\frac{dy}{dx} = 0$, at some point in the domain, then at that point $f(x)$ is neither increasing nor decreasing.

Let us now do an example.

**Example 6 :** Find the range of values of x for which the function $y = x^2 - 4x + 4$ increases with x.

**Solution :** It is required to find the interval in which $\frac{dy}{dx} > 0$. Differentiating the given function with respect to x

$\frac{dy}{dx} = 2x - 4 = 2(x-2)$.

Now, $2(x-2)$ is positive for all $x > 2$, thus, function is increasing.

If $x < 2$ then $\frac{dy}{dx}$ is negative and function decreases, and when $x = 2$, it is stationary.

Therefore, the function increases for all $x > 2$.

You may now do these exercises easily.

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E9) Find the stationary values of $16x - 4x^3$.

E10) Find the range of values of x for which the function $f(x) = (x-3)(x-1)$ is
(a) increasing (b) decreasing (c) has stationary values.

E11) Describe the behaviour of the function $y = \sin x$ in the interval $(0 \leq x \leq \pi)$.

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Thus, you have seen above that the sign of first derivative $\frac{dy}{dx}$ plays an important role in determining the nature of the curve defining a function $y = f(x)$ in a certain domain.

### 7.4.2 Concavity

Let us now study the role of second derivative $\frac{d^2y}{dx^2}$ in determining the behaviour of the curve $y = f(x)$ in a given domain.

Now, what happens if $\frac{d^2y}{dx^2} > 0$, at the point P? This means that $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$, that is, $\frac{dy}{dx}$ is increasing at P. Then, the slope of the graph is increasing at P, since $\frac{dy}{dx}$ is the slope of the graph.

You look at Fig. 4(a) and geometrically try to visualise the general shape of a graph at point P where the slope is increasing. Clearly, the tangent at a point P is turning anti-clockwise as the point moves along the curve from left to right and takes the position $P_1$, $P_2$, $P_3$ and $P_4$. Also the tangent lines at all these points lie below the

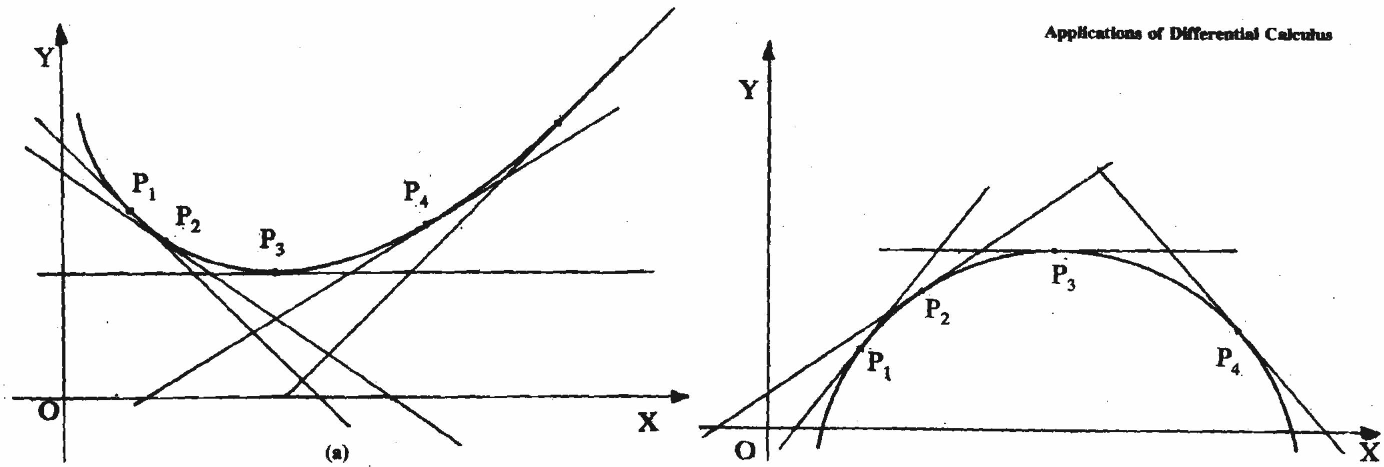


Fig. 4

arc. Such a position or arc of a graph is called "concave upward" and the graph $y = f(x)$ is said to be concave upward at the point P.

Similarly, if $\frac{d^2y}{dx^2} < 0$ at P, then $\frac{dy}{dx}$ is decreasing (Fig. 4(b)). Thus, the slope of the graph is decreasing at P. Therefore, the tangent at a point P is turning clockwise as P moves along the curve from left to right.

Summarising the above discussion, we can now give the following definition.

**Definition :** Given a function $y = f(x)$ defined in some domain.

If $\frac{d^2y}{dx^2} > 0$ in an interval I of the domain, then the curve is said to be **concave upwards** in the interval.

If $\frac{d^2y}{dx^2} < 0$ in the interval I, then the curve is said to be **concave downwards.**

Let us now consider the following example.

**Example 7 :** Examine the graph of the function $y = \frac{1}{x}$ $(x > 0)$ for concavity.

**Solution :** We have,

$\frac{dy}{dx} = -\frac{1}{x^2}$ and $\frac{d^2y}{dx^2} = \frac{2}{x^3}$. Since $x > 0$, $\frac{d^2y}{dx^2} > 0$ for all x and from the above definition we can conclude that the graph of the function turns anti-clockwise and is, therefore, concave upward (see Fig. 5).

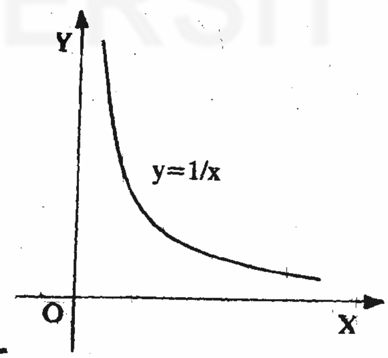


Fig. 5

You may now try this exercise.

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E12) Determine the intervals in which the graph of the following functions are concave.

a) $v = -\frac{1}{3}u^2 + u$

b) $y = x^3 - x$

c) $y = e^x$

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After having discussed the meaning of the signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ we are now in a position to determine the values of a function which are the greatest or the least in their immediate neighbourhood. Such values are known as **maximum** and **minimum** values respectively.

### 7.4.3 Criteria for Extreme Values

A knowledge of maximum and minimum values of a function is of great help in drawing its graph. We now obtain a criteria by which these values can be determined.

We begin with an example.

**Example 7 :** Given $f(x) = \frac{x^3}{3} - x^2 - 3x + 5$; find when f(x) increases, when it decreases and when it is stationary.

**Solution :** We first find $f'(x) = x^2-2x-3 = (x+1)(x-3)$.
It is clear that $f'(x) > 0$ when $x < -1$ or $x > 3$, $f'(x) < 0$ when $-1 < x < 3$ and $f'(x) = 0$ when $x = -1$ or $x = 3$.
If you look at the graph of f(x) (see Fig. 6) you will notice that

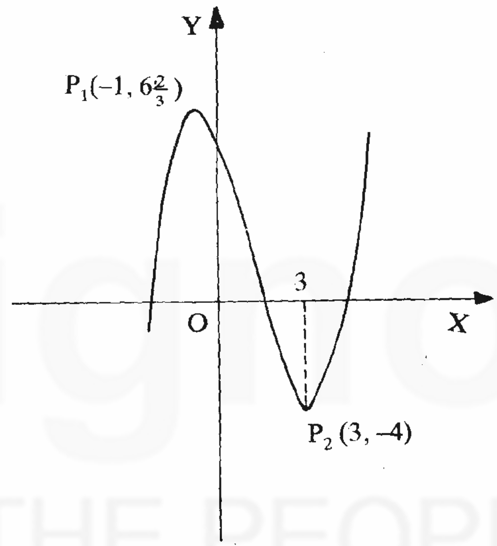


Fig. 6

i) f(x) increases from $x = -\infty$ to $x = -1$

ii) f(x) is stationary at $x = -1$

iii) f(x) decreases from $x = -1$ to $x = 3$

iv) f(x) is stationary at $x = 3$

v) f(x) increases from $x = 3$ to $x = +\infty$

The points $P_1$ and $P_2$ in Fig. 6 are of special interest and importance. The graph is stationary at both these points. Since at $P_1$ the graph stops rising and starts to fall, $P_1$ is called a **maximum point** (point of maxima) of the graph. Similarly, $P_2$ is called a **minimum point** (point of minima) of the graph, since at $P_2$ the graph stops falling and starts to rise. Points $P_1$ and $P_2$ are also called **extreme** points. The value of the function f at $P_1$ is called a maximum value of f(x) and the value of f at $P_2$ is called the minimum value of f(x). Also, the abcissas corresponding to the points $P_1$ and $P_2$ are called the **critical values** of f. In Example 7, the numbers $-1$ and 3 are the critical values of f.
**Note** that a maximum point is not necessarily the highest point of a graph nor a minimum point necessarily the lowest. In fact a graph may have more than one of each. These are called the **local maxima** or **local minima** of a function.

Let us once again consider Example 7 and see what does the geometrical facts mentioned above formulate analytically?
We have, $f'(x) = (x+1)(x-3)$. Thus $f'(x) = 0$ for $x = -1$ or 3. These numbers $-1$ and 3 divide the x-axis into three intervals (see Fig. 7).

I $]-\infty,-1[$, II $]-1,3[$, III $]3,\infty[$


Fig. 7

The sign of $f'(x)$ remains the same throughout in each of these intervals. In I, $f'(x) > 0$; in II, $f'(x) < 0$; and in III, $f'(x) > 0$. This shows that as we go from I to II through $x = -1$, $f'(x)$ is $+$, $0$, $-$, that is, $f(x)$ is respectively increasing, stationary, decreasing, and therefore $f(-1)$ is a maximum.
Similarly, as we go from II to III through $x = 3$, $f'(x)$ is $-$, $0$, $+$ that is, $f(x)$ is respectively decreasing, stationary, increasing, and therefore $f(3)$ is a minimum. In general, given a continuous function $y = f(x)$ with $f'(x)$ finite at every point of the graph, we have the following test for finding maxima and minima.

**Test 1 :** If $f'(x) = 0$ at a point P of the graph and

i) If $f'$ changes sign from positive to negative in crossing the point P, then f has a maximum value at P.

ii) If $f'$ changes sign from negative to positive in crossing the point P, then f has a minimum value at P.

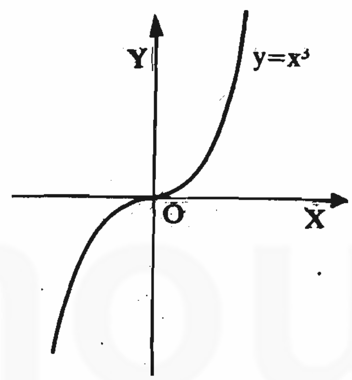


Fig. 8

Let us now consider this example.

**Example 8 :** Test the function $f(x) = x^3$, for maximum and minimum values.

**Solution :** We have, $f'(x) = 3x^2 = 3(x-0)(x-0)$. Thus $f'(x) = 0$ gives $x = 0$ as the only critical value. The point $x = 0$ divides the x-axis into two intervals (see Fig. 8).
I $]-\infty,0[$, II $]0,+\infty[$
If $x < 0$, then $f'(x) > 0$
If $x = 0$, then $f'(x) = 0$
If $x > 0$, then $f'(x) > 0$

Hence $f(x)$ increases for $x < 0$, is stationary at $x = 0$, increases for $x > 0$. Hence $f(x)$ has no maximum or minimum values. The graph $y = x^3$ has no maximum or minimum points but is stationary at $(0,0)$.

So far we have seen the role of sign of $f'(x)$ in obtaining the maximum and minimum values of a function $y = f(x)$. We shall now see how the sign of $f''(x)$ helps in obtaining the maximum and minimum values of $f(x)$

Consider a continuous graph, $y = f(x)$, as in Fig. 9, with $f''(x)$ finite at every point of the graph.

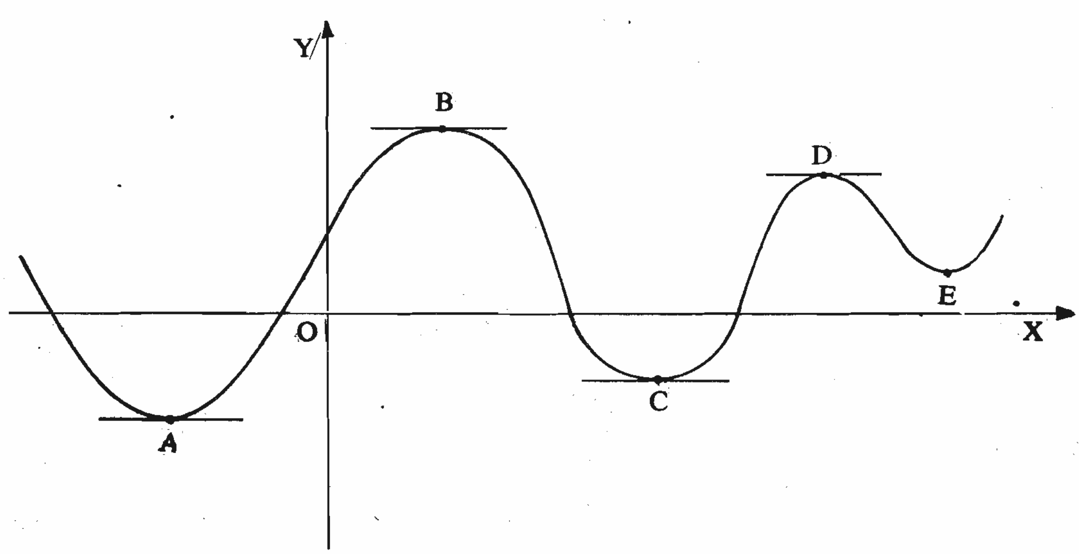


Fig. 9

It is clear that function changes from increasing to decreasing as we cross points B and D. Thus, f′ changes sign from positive to negative. Whereas, at points A, C, E, it changes from a decreasing function to an increasing one and consequently f′ changes sign from negative to positive. Since the graph is continuous $\frac{dy}{dx}$ must vanish at each of the points A, B, C. D and E. In other words, these are the stationary points of f.

Further, you may notice that the function is concave downwards at points which are very close to B and D, and it is concave upwards at points close to A, C and E.

Now at points B and D since $f' = 0$ and while crossing them f′ changes sign from positive to negative, f has a maximum value. Then at these points $f''(x)$ must be negative. For if $f''(x) > 0$, then $f'(x)$ is increasing at these points with increase in x. But, $f'(x) = 0$ at these points so that, $f'(x)$ must have increased from negative value to zero and then to positive value which determines a point of minima. Hence it is a contradiction since f has a maximum value at B and D. Thus, at these points $f''(x) < 0$.

Similarly, at points A, C and E where f′ changes sign from negative to positive and hence are the points of minima, $f''(x) > 0$. We now summarise the above discussion in the form of following test.

**Test 2 :** Let $y = f(x)$ be a continuous function with $f''(x)$ finite at every point of the graph $y = f(x)$. Let $f'(x) = 0$ at a point P of the graph, then

i) If $f''(x) < 0$ at P, then P is a point of maxima.

ii) If $f''(x) > 0$ at P, then P is a point of minima.

The above test for maxima and minima is very useful and easy to apply. In the cases where, $f'(x) = 0$ and also $f''(x) = 0$ at those points, the test cannot be applied. In such cases we go in for Test 1.

Let us now do an example.

**Example 9 :** Find the points of maxima and minima of the function $y = x^3 - 3x$.

**Solution :** Here, $\frac{dy}{dx} = 3x^2 - 3 = 3(x^2-1)$.

For extreme or stationary points we must have $\frac{dy}{dx} = 0$.

Therefore, $x^2-1 = 0$ gives $x = +1$ or $-1$.

Now, $\frac{d^2y}{dx^2} = 6x$.

When $x = 1$, $\frac{d^2y}{dx^2} = 6$ which is positive.

Thus, the function has a minimum at $x = 1$.

When $x = -1$, $\frac{d^2y}{dx^2} = -6$ which is negative.

So, the function has a maximum at $x = -1$.

If we substitute $x = 1$, in the function for y we get,
$y = 1^3 - 3(1) = 1 - 3 = -2$ as the minimum value of f(x). But, you might have noticed that $(-2)$ is by no means the lowest value of the function, because if we put $x = -3$, we get, $y = -27+9 = -18$ and if we put $x = -4$, then $y = -52$ which are much less than $-2$. Similarly $f(-1) = 2$, is not the largest value of f.
Therefore, $x = 1$ is a **local minimum** and $x = -1$ a **local maximum.** They are called local because these values pertain only to the neighbouring points.
The highest maximum or the lowest minimum values of the function on the entire domain of consideration are called **absolute maximum** or the **absolute minimum** respectively.

From the above example it is clear that a **maximum or a minimum value is also a stationary value but a stationary value may neither be a maximum nor a minimum value.** For instance, as we have seen in Example 8, the function $y = x^3$ has $x = 0$ as stationary point but a function does not have either a maximum or a minimum at $x = 0$. In this case $\frac{d^2y}{dx^2} = 6x = 0$ at $x = 0$. Such points are called points of **inflexion.**

Let us now look at another example.

**Example 10 :** Find the greatest and least values of
$y = 3x^4 - 2x^3 - 6x^2 + 6x + 1$ in the interval $[0,2]$.

**Solution :** $\frac{dy}{dx} = 12x^3 - 6x^2 - 12x + 6 = 6(x-1)(x+1)(2x-1)$

Thus, $\frac{dy}{dx} = 0$ for $x = 1, -1, \frac{1}{2}$.

The value $x = -1$ will not be considered because it does not belong to the interval $[0,2]$.

Now, $\frac{d^2y}{dx^2} > 0$ at $x = 1$ and $\frac{d^2y}{dx^2} < 0$ at $x = \frac{1}{2}$.

Thus, $x = 1$ is a point of minima and $x = \frac{1}{2}$ is a point of maxima.

Also, $f(1) = 2, f(\frac{1}{2}) = \frac{39}{16}$.

and $f(0) = 1$, $f(2) = 21$

We notice that the points $x = 1, 1/2$ are the extreme points of the function in the given interval but you may note they are the points of local minima and maxima respectively. In fact, the function over the entire interval $[0,2]$ has least value 1 at $x = 0$ and greatest value 21 at $x = 2$. Hence, these are the points of absolute minima and absolute maxima respectively.
Thus, we make the following remark.

**Remark :** The greatest (maximum) and least (minimum) values of a continuous function $f(x)$ in any interval $[a,b]$ are either $f(a)$ and $f(b)$ or are given by the values of $x$ for which $f'(x) = 0$. In case the function $f$ is increasing in the interval $[a,b]$ and $a < b$, then we can straightaway say that $f(a)$ is the minimum value of $f$ and $f(b)$ is the maximum value of $f$ in the interval $[a,b]$.

In this unit we will concentrate only on local maximum and local minimum thus dropping the term local.

You may now try these exercises.

---

E13) Find the points of maxima or minima of the following functions.

a) $f(x) = 2x^3 - 3x^2 - 12x + 5$

b) $f(x) = 12 + 5x - 2x^2$

c) $f(\theta) = a\cos^2\theta + b\sin^2\theta \ (a > b)$

E14) Find the points of maxima and minima of the function $y = x^4$.

E15) The sum of two numbers is 40. Find the maximum value of their product.

---

After learning the behaviour of a function $y = f(x)$ at points where $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ changes sign, we now look at its behaviour at points which are far away from the origin.

## 7.5 ASYMPTOTES

Let P(x,y) be a point on the graph of a given function $y = f(x)$ and let L be a given non-vertical line. The distance d from P to L is measured along the line through P perpendicular to L (see Fig. 10).

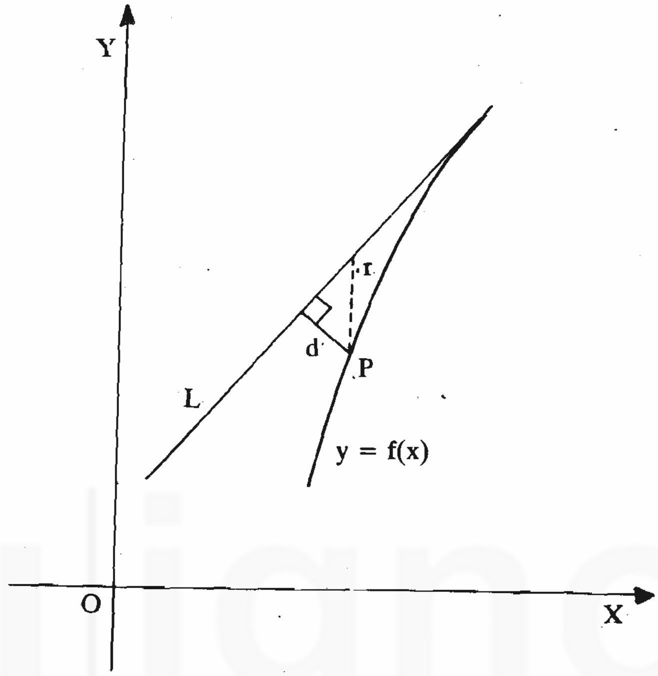


Fig. 10

This line L is said to be an asymptote of the curve if as the point P on the curve tends to infinity along the curve, the perpendicular distance of P from the straight line tends to zero. In other words the line L is an asymptote of the curve if $d \to 0$ as $x \to \infty$ or as $x \to -\infty$. A curve may or may not have an asymptote. For example, you know that for the function, $y = \ln x$, $\ln x \to -\infty$ as $x \to 0$ (as in Fig. 11). In this case we say that the curve approaches the negative y-axis asymptotically.

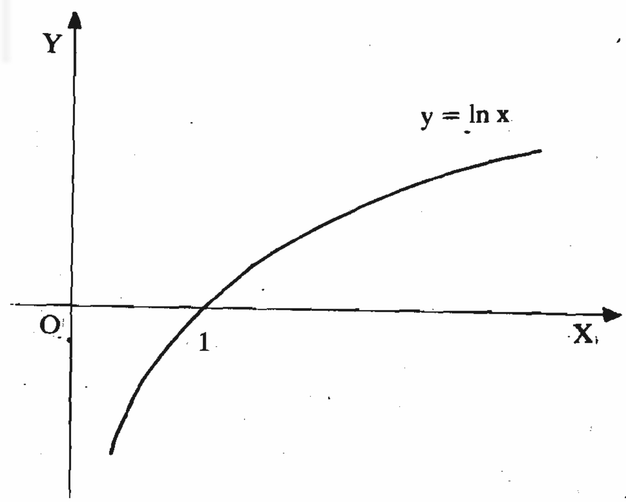


Fig. 11

If the asymptotes of the curve are parallel to either axis, you can spot them by inspection almost immediately.

For example, if $y = \frac{x^2}{x^2+1}$. Here $y \to 1$, as $x \to +\infty$ or $x \to -\infty$.

Therefore $y = 1$ is an asymptote which is parallel to the x-axis (Fig. 12).

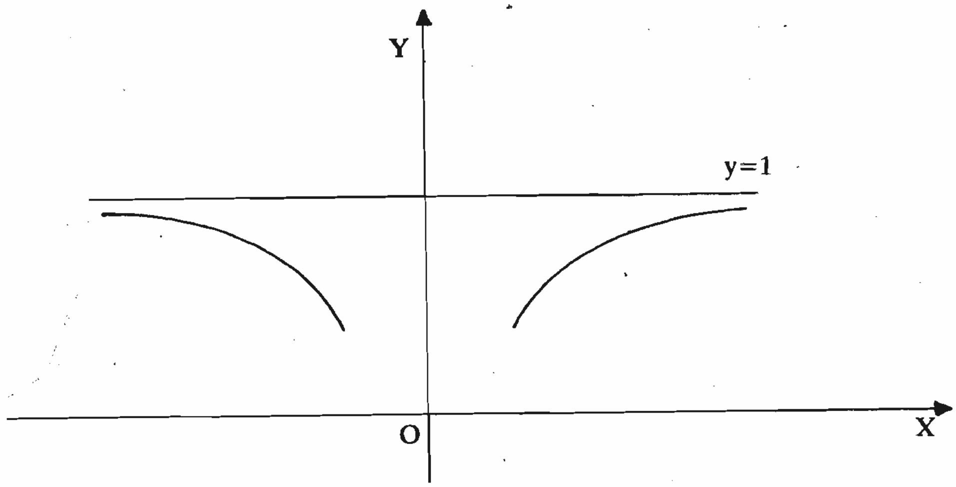


Fig. 12

Take another example:

$y = \frac{3x+2}{x+3}$. Writing it as $y = 3 - \frac{7}{x+3}$, we see that $y \to \infty$, as $x \to -3$. Rewriting the given function as $x = -3 - \frac{7}{y-3}$, we see that $x \to \infty$ as $y \to 3$.

Therefore, $x = -3$ and $y = 3$ are the asymptotes.

You may now try this exercise.

---

E16) Find the asymptotes for the following functions:

a) $y = a - \frac{b}{x}$,

b) $y = 4 + \frac{9}{x+2}$.

---

An easy method of finding asymptotes to a curve when they exist is based on the theory of equations. We shall give only the rules of manipulation.

(1) Arrange the given equation in descending powers of x. Equate to zero the coefficient of the highest power of x provided this is not a constant. Solving the resulting equation, you will get the asymptotes parallel to the x-axis.

For example, consider the function given by $y^2(x^2-a^2) = x$. Write it as $x^2y^2 - x - y^2a^2 = 0$. The highest power of x is 2 and the coefficient of $x^2$ is $y^2$ which is not a constant. Therefore, $y^2 = 0$ that is, $y = 0$ gives the asymptote parallel to the x-axis.

(2) To get the asymptotes parallel to the y-axis, arrange the given equation in descending powers of y and equate to zero the coefficient of the highest power of y, provided it is not a constant.

For example, suppose $y^2(x-b) = x^3 + a^3$. To find the asymptote parallel to the y-axis, we write the function $y^2(x-b) - x^3 - a^3 = 0$. The highest power of y is 2 and the coefficient of $y^2$ is $x-b$. Thus, the equation of the asymptote is $x-b = 0$ or $x = b$.

(3) Asymptotes that are not vertical or horizontal may be more difficult to determine. If you look at Fig. 10, you will observe that the line L is an oblique asymptote to a given curve. Suppose that the equation of the line L is written in the form $y = mx + c$. To find this oblique asymptote, first substitute $mx+c$ for y in the given equation. Then arrange the given equation in descending powers of x and equate to zero the coefficients of two highest powers of x. Solve these two equations for m and c. Substitute the values of m and c in $y = mx + c$ to get the equation of the asymptote.

Consider for example, $y^3 = x^2(x-a)$. Put $y = mx + c$ in the equation to get $(mx+c)^3 = x^2(x-a)$. Coefficients of the two highest powers of x are coefficients of $x^3$ and $x^2$. Equating them to zero we get $m^3 - 1 = 0$ and $3m^2c + a = 0$. On solving the two equations we find that $m = 1$ is the only real root. The corresponding value of c is $c = \frac{a}{3}$. Therefore, the asymptote is $y = x - \frac{a}{3}$.

You can now do the following exercise easily.

---

E17) Find the asymptotes of the following curves.

a) $y = \dfrac{2x-3}{x^2-3x+2}$

b) $y = \dfrac{x^2-2x-8}{x-1}$

c) $y^2(x-1) - x^3 = 0$

d) $y = \dfrac{x}{x^2-1}$.

---

Now, that we have already learnt the methods of finding the maxima, minima of a given function at a point and the equation of an asymptote to a given curve. We are now all set to draw the curve represented by a given equation. The process of drawing a curve is called curve tracing.

## 7.6 CURVE TRACING

The object of curve tracing is to find the general appearance of a curve without plotting lots of points on the graph and avoiding laborious numerical calculations.

For tracing a curve whose equation is given in the rectangular Cartesian coordinate system, that is, in terms of x and y, you must remember the following properties of curves.

**P1:** If the equation remains unchanged when x is replaced by $-x$, then the curve is symmetric about the y-axis.

**P2:** If the equation is unchanged when y is replaced by $-y$, then the curve is symmetric about the x-axis.

**P3:** If the equation remains unaltered when x and y are interchanged, then the curve is symmetric about the line $y = x$. (That is, the line passing through the origin and making an angle of 45° with the positive direction of the x-axis.)

**P4:** If the equation remains unaltered when the signs of both x and y are replaced by their opposites, the curve is symmetric about the origin that is, there is a symmetry in the opposite quadrants.

These four points will help you to find the symmetries of the curve.

After the given function has been tested for symmetries the following steps are to be performed.

1) Find the values of x for which y is not defined (imaginary). The curve does not exist for these values of x.
2) If there is no constant term in the equation, the curve passes through the origin.
3) Find the points where the curve cuts the axes. That is those values of x and y for which the points (x,0) and (0,y) lie on the given curve.
4) Find those values of x for which $y = 0$ or y tends to $\pm \infty$
5) Find the asymptotes to the given curve, if any.

6) Find the points of maxima and minima of the given curve to get an idea about the shape of the curve.

We will now take up a few simple standard cases of curve-tracing.

A) **Witch of Agnesi** (named after the Italian mathematician Maria Gaetna Agnesi (1718–1799)). The equation to the curve is given by

$x^2y = 4a^2(2a-y)$.

i) The curve is symmetrical about the y-axis.

ii) The curve does not pass through the origin, since (0,0) does not satisfy it.

iii) It cuts the y-axis at $y = 2a$ (putting $x = 0$). It does not intersect the x-axis.

iv) $x^2$ is negative for $y > 2a$. Therefore x is imaginary. Hence the curve does not exist for $y > 2a$. Writing $x^2 = 4a^2 \left(\frac{2a-y}{y}\right)$, we see that $x = \pm \infty$ as $y \to 0$. Therefore, $y = 0$ is an asymptote.

v) There is a maxima at $x = 0$ and points of inflexion are

$x = \pm \frac{2a}{\sqrt{3}}$.

vi) When y decreases from 2a to 0, x increases from 0 to $\infty$. Thus, the curve obtained is as in Fig. 13.

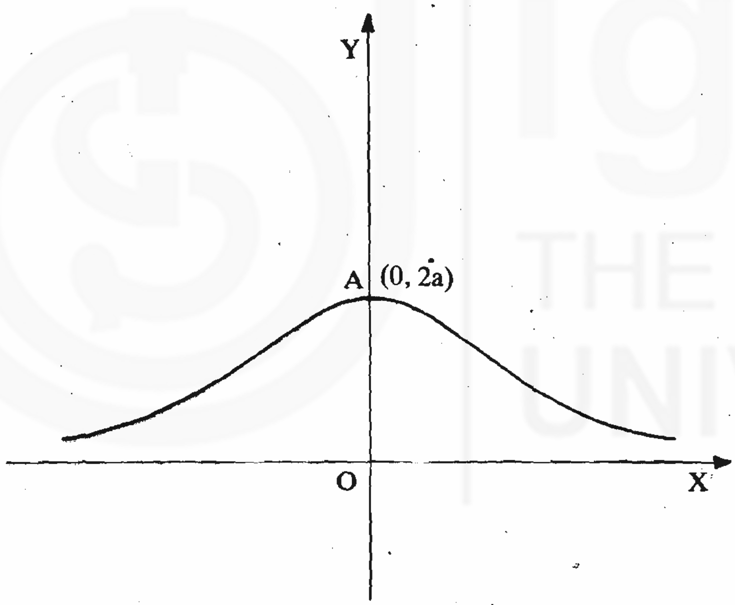


Fig. 13

B) **Cissoid of Diocles** (after the Greek mathematician Diocles around 180 BC). The equation of the curve is

$y^2(2a-x) = x^3$

i) The curve is symmetric about the x-axis.

ii) The curve passes through the origin.

iii) Clearly, $2a-x = 0$ is the asymptote.

iv) The equation of the curve can be written as $y^2 = \frac{x^3}{2a-x}$, which shows that for values of $x > 2a$, y is imaginary that is, the curve does not exist for values of $x > 2a$. Similarly, the curve does not exist for negative values of x. Also as $x \to 2a$, $y \to \infty$.

v) The tangent at the origin is given by $y^2 = 0$ or $y = 0$. Thus, the x-axis is the tangent at the origin.

vi) $\frac{dy}{dx} > 0$ for $x > a$. Therefore, the function is increasing in the semiclosed interval, $[0,a[$.

The tangent at the origin can also be obtained by equating the lowest order term to zero.

The shape of the curve is as shown in Fig. 14.

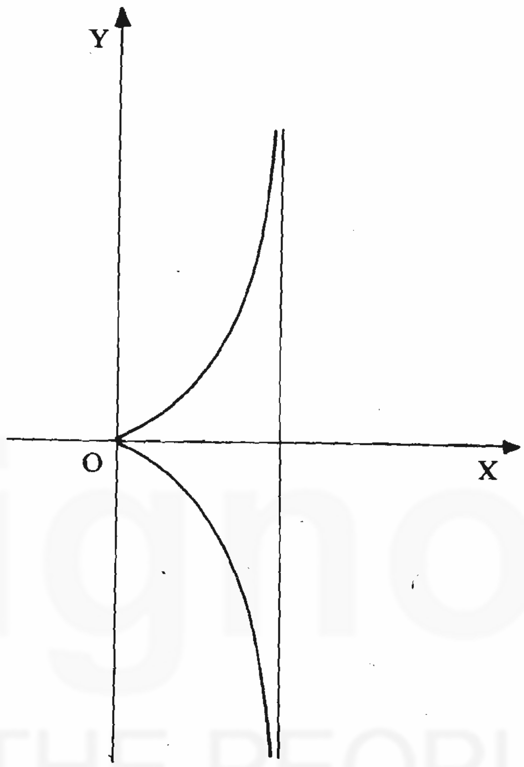


Fig. 14

C) **Cubical parabola**

Consider the curve given by the equation $y = x^3$.

i) If we change y to $-y$ and x to $-x$ the equation of the curve does not change. Hence, there is a symmetry in the opposite quadrant or there is a symmetry about the origin.

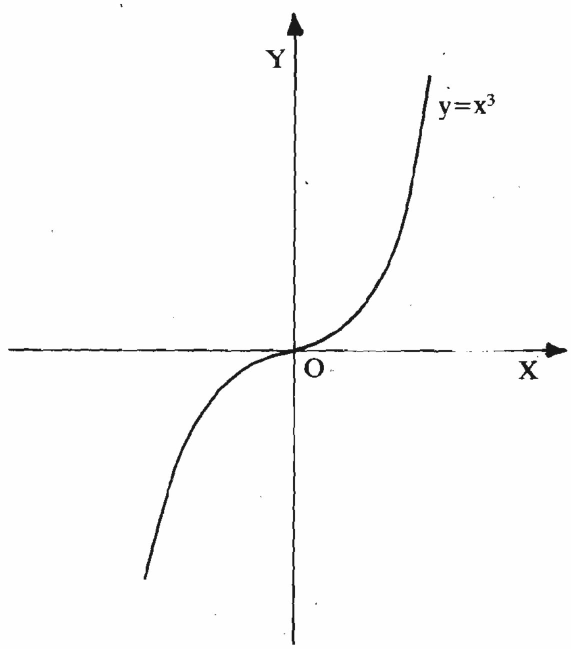


Fig. 15

ii) The curve passes through the origin, the tangent at origin being $y = 0$ (equating the lowest order term to zero).

iii) The curve crosses the axes only at the origin.

iv) As $x$ increases from 0 to $\infty$, $y$ also increases from 0 to $\infty$.

v) There are no asymptotes because the coefficients of both $x^3$ and $y$ are constants.

vi) $\frac{dy}{dx} = 3x^2 = 0$ for $x = 0$ and $\frac{d^2y}{dx^2} = 6x\Big|_{x=0} = 0$. Therefore, origin is a point of inflexion. With the above data the curve is as shown in Fig. 15.

After studying the above examples you must have developed a good understanding about the curve tracing. The following exercise will help you to test your knowledge.

---

E18) Trace the following curves.

a) $y = x^2$

b) $y = e^x$

c) $xy^2 = 4a^2(2a-x)$

---

So far, we have only talked about the functions of the form $y = f(x)$, that is, about the functions involving only one independent variable namely, $x$. In the next section, we extend our consideration to functions of two independent variables.

## 7.7 FUNCTIONS OF TWO VARIABLES

As we go ahead and define various concepts involving functions with two variables, you will notice that most of the concepts are easy generalisations of those for functions of one variable. We shall be discussing these concepts in brief. To start with we give the following definition.

**Definition :** A variable $z$ is said to be a **function of two independent variables** $x$ and $y$, if for each set of values of $(x,y)$ we can determine a value of $z$, so that there is a correspondence between $z$ and the pair $(x,y)$. We denote this correspondence by the notation.

$z = f(x,y)$ or $z = z(x,y)$.

The domain of the function is the set

$\mathbf{R}^2 = \{ (x,y) : x \in \mathbf{R}, y \in \mathbf{R} \}$ or a subset of $\mathbf{R}^2$ on which the function f is defined and real.

For example, the domain of $f(x,y) = 3x+5y$ is $\mathbf{R}^2$. The domain of $T(x,y) = (xy)^{1/2}$ is only positive $x$ and positive $y$ or only negative $x$ and negative $y$. If only one of $x$ or $y$ is negative, $T(x,y)$ will no longer remain real. Similarly, the domain of $f(x,y) = \sqrt{8-(3x+2y)}$ is the set of all $(x,y)$ such that $3x+2y < 8$. Just as in the case of functions of a single variable, the concepts of limit and continuity are closely related for a function of two variables also. We shall study them now.

### 7.7.1 Limit of a Function of Two Variables

You may find it useful to compare the definition given here with the definition given in Unit 6, for the limit of a function of one variable and to note how the generalisation has been made.

**Definition :** A function $f(x,y)$ is said to tend to a limit L as $x \to a$ and $y \to b$, if as $x$ approaches a and $y$ approaches b, the function $f(x,y)$ gets closer and closer to L and we write $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y) = L$

**Note** that for a continuous function the lim f(x,y) as (x,y) approaches (a,b) is the same as lim f(x,y) as first x approaches a and then y approaches b. Mathematically, we express it as $\lim\limits_{y \to b} \lim\limits_{x \to a} f(x,y)$ or $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y)$. Similarly, first y can approach b and then x approaches a, that is $\lim\limits_{x \to a} \lim\limits_{y \to b} f(x,y)$ or $\lim\limits_{\substack{y \to b \\ x \to a}} f(x,y)$. But, the limit L exists only if the above two limits are equal.

For example, if $f(x,y) = \dfrac{2x}{x^2+y^2+1}$ then

$$\lim_{y \to 0} \{ \lim_{x \to 0} f(x,y) \} = \lim_{y \to 0} \{0\} = 0 = \lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) \text{ and}$$

$$\lim_{x \to 0} \{ \lim_{y \to 0} f(x,y) \} = \lim_{x \to 0} \left\{ \frac{2x}{x^2+1} \right\} = \frac{0}{1} = 0 = \lim_{\substack{y \to 0 \\ x \to 0}} f(x,y)$$

$$\text{Similarly, } \lim_{\substack{x \to 1 \\ y \to 2}} f(x,y) = \frac{2}{6} = \frac{1}{3} = \lim_{\substack{y \to 2 \\ x \to 1}} f(x,y)$$

Hence the function f(x,y) has the limit 0 and 1/3 at points (0,0) and (1,2) respectively.

Remember that while taking the $\lim\limits_{x \to a} f(x,y)$, y is kept constant. Similarly, while taking $\lim\limits_{y \to b} f(x,y)$, x is kept constant.

We emphasise once again that the concept of limit does not depend upon the value of f(a,b) or even upon whether f is defined at (a,b).

We now come to the continuity of f(x,y).

### 7.7.2 Continuity of a Function of Two Variables

Consider the function defined as follows:

$$f(x,y) = x^2 + y, \text{ when } (x,y) \neq (1,1)$$
$$= 0, \text{ when } (x,y) = (1,1).$$

You can easily see that the limit of the function as $x \to 1$ and $y \to 1$ is 2. But the value of the function when x = 1 and y = 1 is defined as zero, which means that at (1,1) the limit of the function is not equal to the given value of the function. In such cases we say that the function is discontinuous at (1,1). We now give the following definition.

**Definition** : A function f(x,y) is said to be **continuous** at (a,b) if the following conditions are satisfied:

i) f(x,y) is defined at (a,b)

ii) $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y) = L$ exists

iii) f(a,b) = L that is, the limit of the function is equal to the value of the function at (a,b).

Some examples of continuous functions are the following:

i) Polynomial functions, which are functions formed as finite linear combinations of non-negative integral powers of the independent variables, are continuous. For example, $f(x,y) = 3 + 2xy + 2x^2 - 3y^2$ is continuous for all x and y.

ii) Functions, which are quotients of polynomial functions are continuous except at zeros of the denominator. For instance, the function $\dfrac{x}{2x+3y}$ is continuous everywhere except at (0,0). At (0,0) it is not at all defined.

iii) Functions obtained by taking roots, trigonometric functions, exponential functions and logarithmic functions are continuous except at the points where the functions are not defined. Some examples of continuous functions are :

a) $f(x,y) = \sqrt{4-(x^2+y^2)}$, $x^2+y^2 \leq 4$.

b) $g(x,y) = \sin \dfrac{xy}{\sqrt{x^2+y^2}}$, $(x,y) \neq (0,0)$.

c) $h(x,y) = e^{x^2}+y^2$, for all $(x,y)$.

d) $\phi(x,y) = \ln |x-y|$, $y \neq x$.

Before we go on to next subsection how about doing an exercise?

---

E19) Determine the domain in which the following functions are continuous

a) $f(x,y) = \sqrt{9 - 2(x^2+y^2)}$

b) $g(x,y) = 6x^3 - 7y^3 + 4xy$ c) $\phi(x,y) = \ln xy$

---

In the next section we will extend the computational rules developed so far for finding derivatives of functions of two variables. The derivative of a function f, of two variables with respect to one of the variables while the other one is kept constant is called the partial derivative of f.

### 7.7.3 Partial Derivatives

You will recall that in the case of a function of one variable $y = f(x)$, the derivative of f with respect to x was the instantaneous rate of change of f with respect to x and was denoted by $\frac{df}{dx}$ or $\frac{dy}{dx}$.

In the same manner, the partial derivative of a function of two variables $z = f(x,y)$ with respect to one of the independent variables, can be regarded as the instantaneous rate of change of z with respect to that variable when the other independent variable is held constant. We describe it as follows:

Let a small change $\delta x$ in x result in a change $\delta z$ in z, keeping y constant. Since $z = f(x,y)$, we can write

$z + \delta z = f(x+\delta x, y)$ and

$\delta z = f(x+\delta x, y) - z = f(x+\delta x, y) - f(x,y)$

Therefore, $\dfrac{\delta z}{\delta x} = \dfrac{f(x+\delta x, y) - f(x,y)}{\delta x}$.

If this quotient $\frac{\delta z}{\delta x}$ tends to a limit as $\delta x \to 0$, then this limit is called the **partial differential coefficient** or the **partial derivative of z with respect to** x and is denoted by $\frac{\partial z}{\partial x}$. It is also denoted by $f_x(x,y)$ or $\frac{\partial f}{\partial x}$ or simply $f_x$. The derivative evaluated at any point (a,b) in the domain of the function is denoted by $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$ or $f_x(a,b)$.

Similarly, keeping x as constant, we can define the partial derivative of f(x,y) with respect to y as

$$\frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial x} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

This is also denoted by $f_y$ or $f_y(x,y)$.

You can see that $\frac{\partial f}{\partial x}$ is simply the ordinary derivative of f with respect to x, keeping y constant and $\frac{\partial f}{\partial y}$ is the ordinary derivative of f with respect to y, keeping x constant. The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are again functions of x and y.

For example, if $z = 2x^2-5xy+4y^2$, then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial x}(5xy) + \frac{\partial}{\partial x}(4y^2) = 4x-5y+0 = 4x - 5y.$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(2x^2) - \frac{\partial}{\partial y}(5xy) + \frac{\partial}{\partial y}(4y^2) = 0-5x+8y = 8y - 5x.$$

You may now try the following exercise.

---

E20) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, where z is given by

a) $z(x,y) = x^4 + 3y^3$
b) $z(x,y) = x \cos y + y \cos x$
c) $z(x,y) = \sin(3x+y)$
d) $z(x,y) = \tan^{-1}(x+y)$
e) $z(x,y) = e^{x-y}$
f) $z(x,y) = x \ln y$
g) $z(x,y) = (x-2y)^{-1}$

---

In Unit 6 we have already learnt the method of obtaining derivative of an implicit function of one variable. We now extend the same ideas to obtain the partial derivatives of implicit functions of two variables.

### 7.7.4 Partial Derivatives of Implicit Functions

So far, all the functions f(x,y) considered were defined explicitly in the form $z = f(x,y)$. The functions of two variables x and y can also be given by an implicit function in x,y,z of the form $xy + yz + zx = 1$. If now we are required to find $\frac{\partial z}{\partial x}$, then we differentiate the given equation partially with respect to x keeping in mind that z is a function of both x and y.

So, $\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial x}(zx) = \frac{\partial}{\partial x}$ (1)

or $y + y\frac{\partial z}{\partial x} + z + \frac{\partial z}{\partial x}x = 0$

or, $\frac{\partial z}{\partial x}(y+x) = -(y+z)$

and, $\frac{\partial z}{\partial x} = -\frac{y+z}{y+x}$ is the partial derivative of z with respect to x.

Similarly, we get $\frac{\partial z}{\partial y} = -\frac{x+z}{y+x}$.

How about trying an exercise now?

---

E21) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if

a) $x^2 + y^2 + z^2 = 16$,
b) $x^2 + 3xy - 2y^2 + 3xz + z^2 = 0$.

---

We now obtain the chain rule for the partial derivatives of composite functions.

### 7.7.5 Partial Derivatives of Composite Functions

We begin by the definition of composite function of two variables.

**Definition :** If z is a function of independent variables x and y, where x and y, in turn are themselves functions of a variable t, then z is called a **composite function** of t. For instance, if

$z = f(x,y)$, where $x = \phi(t)$ and $y = \psi(t)$

then, $z = f[\phi(t), \psi(t)]$.

You will realise that since z is directly a function of t now, you can find $\frac{dz}{dt}$ using rules for ordinary differentiation. However, a rule for obtaining $\frac{dz}{dt}$ will now be established which can be applied more conveniently to difficult cases also.

As before, let $\delta x$ and $\delta y$ be the small changes in x and y respectively, corresponding to a change $\delta t$ in t and let $\delta z$ be the consequent change in z. Then,

$z + \delta z = f(x+\delta x, y+\delta y)$

and $\delta z = f(x+\delta x, y+\delta y) - f(x,y)$

Therefore, $\frac{\delta z}{\delta t} = \frac{f(x+\delta x, y+\delta y) - f(x,y)}{\delta t}$

which can be re-written as

$$\frac{\delta z}{\delta t} = \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x,y+\delta y) - f(x,y)}{\delta t}$$

(we have added and subtracted $f(x, y+\delta y)$)

$$\Rightarrow \frac{\delta z}{\delta t} = \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y+\delta y) - f(x,y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

Note that when $\delta t \to 0$, both $\delta x \to 0$ and $\delta y \to 0$.

$$\text{Thus, } \frac{dz}{dt} = \lim_{\delta t\to 0} \frac{\delta z}{\delta t} = \lim_{\delta x\to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \lim_{\delta t\to 0} \frac{\delta x}{\delta t}$$

$$+ \lim_{\delta y\to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} \lim_{\delta t\to 0} \frac{\delta y}{\delta t}$$

Here, we have used the fact that limit of the product = product of the limits.

Hence, we obtain that $\frac{dz}{dt} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

**This rule is called the chain rule for differentiating composite functions.**

Let us now use this rule in the following example for finding the derivative of the given function.

**Example 11 :** Find $\frac{dz}{dt}$ for $z = x^2y + xy^2$ where $x = at^2$, $y = 2at$.

**Solution :** We have $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

Now, $\frac{\partial z}{\partial x} = 2xy + y^2$ ; $\frac{\partial z}{\partial y} = x^2 + 2xy$ ; $\frac{dx}{dt} = 2at$ ; $\frac{dy}{dt} = 2a$.

On substitution $\frac{dz}{dt}$ becomes

$\frac{dz}{dt} = (2xy + y^2)2at + (x^2+2xy)2a.$

Putting $x = at^2$, $y = 2at$,

$\frac{dz}{dt} = 2at\ (2\cdot at^2\cdot 2at + 4a^2t^2) + 2a(a^2t^4 + 2\cdot at^2\cdot 2at).$

Therefore, $\frac{dz}{dt} = a^3(16t^3 + 10t^4).$

Here we can obtain $\frac{dz}{dt}$ in another way also. Substituting $x = at^2$

and $y = 2at$ in $z = x^2y + xy^2$, we get

$$z = a^2t^4 \cdot 2at + at^2 \cdot 4a^2t^2$$
$$= 2a^3t^5 + 4a^3t^4.$$

Thus, $\frac{dz}{dt} = 10a^3t^4 + 16a^3t^3$.

You can try this exercise now.

---

E22) Find $\frac{dz}{dt}$ in the following cases.

a) $z = x^5y^4$, where $x = t^2$ and $y = t^3$

b) $z = e^{xy^2}$, where $x = t\cos t$, $y = t\sin t$

c) $z = x^2 + 3xy + 5y^2$, where $x \cos t$, $y = \sin t$

d) $z = \ln (x^2+y^2)$, where $x = e^{-t}$ and $y = e^t$

---

Now suppose that $z = f(x,y)$ and x and y are functions not of one variable t, as above, but two variables say, u and v, possessing first order partial derivatives. In other words $z = f(x,y)$ where $x = \phi(u,v)$ and $y = \psi(u,v)$.

In order to find a rule for obtaining $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$, we use the technique of partial differentiation and regard v as a constant while obtaining $\frac{\partial z}{\partial u}$ and regard u as a constant while getting $\frac{\partial z}{\partial v}$ and apply the previous chain rule.

Thus, $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

and, $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$

Consider for example, $u = x^3-xy+y^3$ where $x = r\cos\theta$ and $y = r\sin\theta$.

Then, $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

Therefore,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = (3x^2-y)\cos\theta + (3y^2-x)\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = (3x^2-y)(-r\sin\theta) + (3y^2-x)(r\cos\theta).$$

The result can also be verified by direct substitution for x and y into u and then differentiating partially. At this stage it will not be difficult for you to understand the concept of the total differential of a function.

**The total differential df of a function f(x,y) is defined as**

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

or $df = f_x dx + f_y dy$

For instance, if $z = x^2y - 3y$,

then, $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 2xy\,dx + (x^2-3)dy$

since $\frac{\partial z}{\partial x} = 2xy$ and $\frac{\partial z}{\partial y} = x^2 - 3$.

This has applications in finding the error in a function of two variables, when the errors in the independent variables are known.

You can now solve the following exercises easily.

---

E23) Find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$, if

a) $u = x^2-xy+y^2$, where $x = r\cos\theta$, $y = r\sin\theta$

b) $u = x^2-y^2$, where $x = 2r-3\theta+4$, $y = -r+8\theta+5$

E24) Find df, where

a) $f(x,y) = x^3e^{x/y}$

b) $f(x,y) = x^3y-4xy^2+8y^3$

---

Recall that in Sec. 7.3, we obtained the second and higher order derivatives for the functions with one variable. Now in the next section we shall obtain the second order derivatives for a function of two variables.

### 7.7.6 Partial Derivatives of Order Two

You know from earlier discussion that if f(x,y) has partial derivatives at each point of its domain, then $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are again functions of x and y. These derived functions may also have partial derivatives.

The partial derivative of $\frac{\partial f}{\partial x}$ with respect to x that is, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, is called the second order partial derivative of f(x,y) with respect to x and written as $\frac{\partial^2 f}{\partial x^2}$ or $f_{xx}$.

Similarly, the partial derivative of $\frac{\partial f}{\partial y}$ with respect to y is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

There are two more partial derivatives of f of second order. These are

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \cdot \partial x} \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \cdot \partial y}$$

We also denote $\frac{\partial^2 f}{\partial y \cdot \partial x}$ by $f_{yx}$ and $\frac{\partial^2 f}{\partial x \cdot \partial y}$ by $f_{xy}$.

These are called mixed derivatives of second order and distinguished by the order in which f(x,y) is successively differentiated with respect to the independent variables. Thus, in $f_{xy}$ we differentiate first w.r.t. y and then w.r.t. x, whereas in $f_{yx}$ we first differentiate first w.r.t. x and then w.r.t. y. If f(x,y) and its partial derivatives are continuous then the order of differentiation is immaterial and, in that case we have $f_{yx} = f_{xy}$.

We now calculate all the second order partial derivatives in the following example.

**Example 12 :** If $f(x,y) = x^2\sin y$, find all the second order partial derivatives.

**Solution :** $f_x = 2x\sin y$, $f_y = x^2\cos y$

$f_{xx} = 2\sin y$, $f_{yy} = -x^2\sin y$

$$f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(x^2\cos y) = 2x\cos y$$

$$f_{yx} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(2x\sin y) = 2x\cos y$$

You notice that $f_{xy} = f_{yx}$.

You may now try these exercises.

E25) Find all the second order partial derivatives of the following functions.

a) $z = x^2 + 5xy + y^2$

b) $f(x,y) = xe^{ay}$

E26) Evaluate all the second order partial derivatives at $(\frac{\pi}{4}, \frac{\pi}{4})$ of

$U(x,y) = x\cos y - y\cos x$

E27) For the following functions, show that $f_{xy} = f_{yx}$.

a) $f(x,y) = x\ln y$

b) $f(x,y) = ax^n y^n$

We shall now be proving a theorem due to Euler.

## 7.8 EULER'S THEOREM

Since the theorem is true only for homogeneous function we shall first begin with the definition of homogeneous function.

Ordinarily, $z = f(x,y)$ is said to be a **homogeneous function of degree n** for some constant n, if the degree of each of its terms in x and y is equal to n. Thus,

$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_{n-1}xy^{n-1} + a_ny^n$ ........ (1)

is a homogeneous function of degree n. This definition of homogeneity applies to polynomial functions only. To enlarge the concept of homogeneity we say that z is a homogeneous function of degree n, if it is expressible as $x^n f(y/x)$. The polynomial function (1) which can be written as

$x^n [a_0 + a_1 (\frac{y}{x}) + a_2 (\frac{y}{x})^2 + \ldots + a_n (\frac{y}{x})^n]$.

is a homogeneous function of degree n according to the second definition also.

We give some more examples below.

**Example 13 :** Show that function $z = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$ is homogeneous.

What is the degree of the function?

**Solution :** Rewrite the given function as a function of $(\frac{y}{x})$. We get,

$$z = \frac{x^{1/4}(1 + (\frac{y}{x})^{1/4})}{x^{1/5}(1 + (\frac{y}{x})^{1/5})} = x^{1/4 - 1/5} f(\frac{y}{x})$$

$$= x^{1/20} f(\frac{y}{x}).$$

Therefore, z is a homogeneous function of degree $\frac{1}{20}$.

Now if you have understood the definition of homogeneous functions, you can do the following exercise easily.

E28) a) Show that $U = \dfrac{x^2(x^2-y^2)^3}{(x^2+y^2)^3}$ is a homogeneous function. What is the degree of this function?

b) Is $z = x^n \ln \frac{y}{x}$ a homogeneous function?

We now give you the statement and proof of Euler's theorem.

**Theorem :** If $z = f(x,y)$ be a homogeneous function of degree n in x and y then, $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$.

**Proof :** Since z is a homogeneous function of degree n by definition,
$z = x^n f(\frac{y}{x})$

Therefore, $\frac{\partial z}{\partial x} = n x^{n-1} f(\frac{y}{x}) + x^n \cdot f'(\frac{y}{x}) \cdot (\frac{-y}{x^2})$ where $f'(\frac{y}{x})$ is the differential coefficient of $f(\frac{y}{x})$ with respect to $(\frac{y}{x})$ and $(\frac{-y}{x^2})$ is the differential coefficient of $\frac{y}{x}$ with respect to x.

Thus, $\frac{\partial z}{\partial x} = nx^{n-1} f(\frac{y}{x}) - yx^{n-2} f'(\frac{y}{x})$

Similarly, $\frac{\partial z}{\partial y} = x^n f'(\frac{y}{x})(\frac{1}{x}) = x^{n-1} f'(\frac{y}{x})$

and $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nx^n f(\frac{y}{x}) - yx^{n-1} f'(\frac{y}{x}) + x^{n-1} y f'(\frac{y}{x})$

$$= nx^n f(\frac{y}{x}) = nz$$

Hence, $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$.

**Example 14 :** Verify Euler's theorem for the function
$f(x,y) = x^4y^2 \sin^{-1}(\frac{y}{x})$.

**Solution :** Rewriting the given function in the form $f(x,y) = x^6 (\frac{y}{x})^2 \sin^{-1}(\frac{y}{x})$, we see that f(x,y) is a homogeneous function of degree 6. Thus, to verify Euler's theorem we have to show that,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 6f.$$

Now, $\frac{\partial f}{\partial x} = 4x^3y^2 \sin^{-1}(\frac{y}{x}) + x^4y^2 \frac{1}{\sqrt{1-(\frac{y}{x})^2}} (\frac{-y}{x^2})$

$$= 4x^3y^2 \sin^{-1}(\frac{y}{x}) - \frac{x^3y^3}{\sqrt{x^2-y^2}}$$

$$\frac{\partial f}{\partial y} = 2x^4y \sin^{-1}(\frac{y}{x}) + x^4y^2 \frac{1}{\sqrt{1-(\frac{y}{x})^2}} (\frac{1}{x})$$

$$= 2x^4y \sin^{-1}(\frac{y}{x}) + \frac{x^4y^2}{\sqrt{x^2-y^2}}$$

and

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 4x^4y^2 \sin^{-1}(\frac{y}{x}) - \frac{x^4y^3}{\sqrt{x^2-y^2}} + 2x^4y^3 \sin^{-1}(\frac{y}{x}) + \frac{x^4y^3}{\sqrt{x^2-y^2}}$$

$$= 6x^4y^2 \sin^{-1}(\frac{y}{x}) = 6f.$$

Thus, Euler's theorem is verified.

You may now try the following exercises.

E29) Verify Euler's theorem for the function

$f(x,y) = ax^2+2hxy+by^2$

E30) If $U = \sin^{-1} \left(\frac{x^2+y^2}{x+y}\right)$, show that

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = \tan U.$$

We now conclude this unit by giving a summary of what we have done in it.

## 7.9 SUMMARY

In this unit we have covered the following points.

1) Equations of tangent and normal to a given curve at any given point.
2) Method of calculating second, third and higher order derivatives of a given function of one variable, and in simple cases, guessing the $n^{th}$ order derivative.
3) Expansion in power series of functions, using Maclaurin's formula.
4) Methods of finding maxima and minima of a given function.
5) Method of tracing a few simple curves and finding their asymptotes, if any.
6) Limit, continuity and partial derivatives upto order two of functions of two variables.
7) Euler's theorem.

## 7.10 SOLUTIONS/ANSWERS

E 1) Tangent is $2x - y + 1 = 0$ and normal is $x + 2y = 7$.

E 2) (a) Tangent is $xx_1 + yy_1 = a^2$ and normal $x_1y - y_1 x = 0$
(b) Tangent is $x_1y + y_1x = k^2$ and normal $xx_1yy_1 - x_1^2 - y_1^2$.

E 3) At $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$. Tangent is $x - 2y + 4a = 0$

Normal is $y + 2x = 12a$.

E 4) $\frac{dy}{dx} = -\frac{b}{a} \tan \theta$. Tangent is $y - \frac{b}{\sqrt{2}} = \frac{-b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$

Normal is $y - \frac{b}{\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{\sqrt{2}}\right)$

E 5) (a) $f'(x) = \frac{1}{2}(x+1)^{-1/2}$; $f''(x) = \frac{-1}{2}\left(\frac{1}{2}\right)(x+1)^{-3/2}$;

$f'''(x) = \frac{1.3}{2^9}(x+1)^{-5/2}$.

(b) $y' = \frac{2a}{y} = \sqrt{a}x^{-1/2}$; $y'' = -\frac{1}{2}\sqrt{a}\, x^{-3/2}$;

$y''' = \frac{1.3}{2^2}x^{-5/2}$

(c) $y' = \cos x$ ; $y'' = -\sin x$ ; $y'' = -\cos x$.

(d) $y' = x + 2x\ln x$ ; $y'' = 3 + 2\ln x$ ; $y'' = \frac{2}{x}$

E 6) (a) $(-1)^n\, n!\, (x+a)^{-n}$;
(b) $(-1)^{n-1}\, (n-1)!\, a^n\, (ax+b)^{-n}$;
(c) $a^{-n} \cos\left(\frac{n\pi}{2} + x\right)$

E 7) (a) $f(0) = e^0 = 1$ and all successive derivatives at $x = 0$ remain 1. Hence the result.

(b) $f(0) = 1$; $f^{(1)}(0) = 0$; $f^{(2)}(0) = -1$ and so on.

Thus, by the substitution in the Maclaurins expansion the result follows.

E 8) (a) $f^{(1)}(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots\ldots$ by simple division, therefore

$f^{(2)}(0) = 0$; $f^{(3)}(0) = -2$.

Therefore $\tan^{-1}x = \frac{2}{3!}x^3$.

(b) $f(x) = \ln(1+x)$, $f^{(1)}(x) = \frac{1}{1+x}$, $f^{(2)}(x) = -\frac{1}{(1+x)^2}$; $f^{(3)}(x) = \frac{2}{(1+x)^3}$

$$= 2\left[1 + (-3)x + \frac{(-3)(-4)}{2!}x^2 + \frac{(-3)(-4)(-5)}{3!}x^3 + \ldots\ldots\right]$$

(using Binomial expansion)

$\therefore f^{(3)}(x) = 2\,[1 - 3x + 6x^2 - 10x^3 + \ldots\ldots]$

E 9) $f'(x) = 16 - 12x^{(2)}$. Stationary values are $\pm\frac{2}{\sqrt{3}}$.

E 10) $f'(x) = 2(x-2)$ (a) Increasing for $x > 2$.
(b) Decreasing for $x < 2$ (c) Stationary at $x = 2$.

E 11) $\frac{dy}{dx} > 0$ in $0 < x < \frac{\pi}{2}$ hence increasing.

$\frac{dy}{dx} < 0$ in $\frac{\pi}{2} < x < \pi$ hence decreasing.

$\frac{dy}{dx} = 0$ at $x = \frac{\pi}{2}$ therefore stationary.

E 12) (a) $y'' = \frac{-2}{3}$ concave downwards for all x.

(b) $y'' = 6x$. Thus curve is concave upwards for $x > 0$, downwards for $x < 0$.

(c) $y'' = e^x$, concave upwards for all x.

E 13) (a) Max at $x = -1$, min at $x = 2$.

(b) Max at $x = \frac{5}{4}$.

(c) Min at $\theta = 0$, and $\theta = \pm 2n\pi$.

E 14) $\frac{dy}{dx} = 4x^3$ thus $x = 0$ is a stationary point.

Now, $\frac{dy}{dx} < 0$ for $x < 0$ and $\frac{dy}{dx} > 0$ for $x > 0$.

Thus $x = 0$ is a point of minima.

E 15) Let x and y be two numbers. Then $x + y = 40$ and $xy = p = x(40 - x)$,

$\frac{dP}{dx} = 0$ when $x = 20$ and $\frac{d^2P}{dx^2} < 0$.

Thus, P is maximum when $x = 20$. Also when $x = 20$, $y = 20$ and therefore maximum $P = 400$.

E 16) (a) Coefficient of y is x and coefficient of x is $(y - a)$ so $x = 0$ and $y = a$ are the asymptotes.

(b) $y \to \infty$ for $x \to -2$. Also, $x = -2 + \frac{9}{y-4}$ and $x \to \infty$ for $y \to 4$. Thus $x = -2$ and $y = 4$ are the asymptotes.

E 17) (a) Coefficient of $x^2$ when arranged in descending powers of x is y. $y = 0$ is an asymptote.
Similarly, $x = 1$ and $x = 2$ are the other asymptotes.

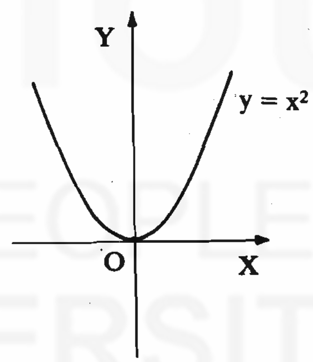

Fig. 16

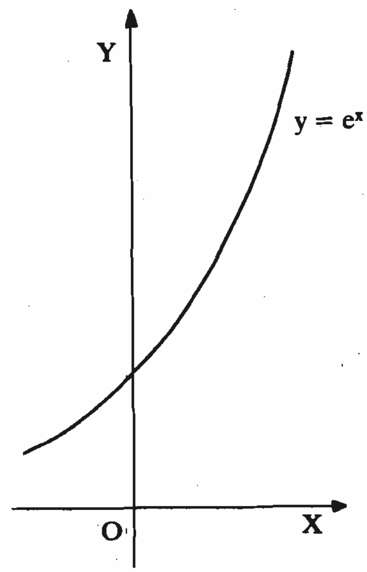

Fig. 17

(b) Substitute $y = mx + c$, put the coefficient of two highest powers x as zero and get $y = x - 1$ as one asymptote.

(c) Coefficient of $y^2$ is $(x-1)$ thus $x = 1$ is one asymptote. Other two oblique asymptotes are $y = x + \frac{1}{2}$ and $y = -x - \frac{1}{2}$.

(d) $x = 1$, $x = -1$, $y = 0$ are the three asymptotes.

E 18) (a) (1) Symmetry about y-axis.

(2) (0, 0) lies on the curve.

(3) y increases from 0 to $\infty$ whenever x increases from 0 to $\infty$ or decreases from 0 to $-\infty$.

(4) $\frac{dy}{dx} = 2x \Rightarrow x = 0$ is a stationary point. Also

$\frac{d^2y}{dx^2} = 2 > 0$, therefore $x = 0$ is a point of minima.

The graph of the function obtained is as shown in Fig. 16.

(b) (1) Curve crosses the y-axis at (0, 1).

(2) As x increases from $-\infty$ to $\infty$, y increases from 0 to $\infty$. The graph of the function is given by Fig. 17.

(c) (1) Symmetry about x-axis.

(2) Curve crosses the x-axis at (2a, 0).

(3) y-axis is an asymptote.

(4) Curve does not exists for $x > 2a$ and $x < 0$.

(5) As x decreases from 2a to 0, y increases from 0 to $\infty$.

The graph of the function is as shown in Fig. 18.

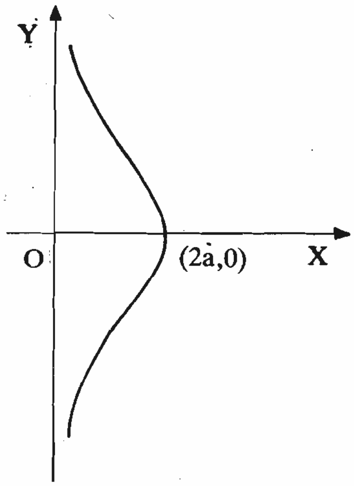


Fig. 18

E 19) (a) $x^2 + y^2 \leq \frac{9}{2}$.

(b) $-\infty < x < \infty$, $-\infty < y < \infty$.

(c) $xy > 0$.

E 20) (a) $z_x = 4x^3$ and $z_y = 6y$.

(b) $z_x = \cos y - y\sin x$ ; $z_y = -x\sin y + \cos x$

(c) $z_x = \cos(3x + y) .3 = 3\cos(3x + y)$

$z_y = \cos(3x+y) .1 = \cos(3x + y)$.

(d) $z_x = \dfrac{1}{1 + (x+y)^2}$ $z_y = \dfrac{1}{1 + (x+y)^2}$

(e) $z_x = e^{x-y}$; $z_y = -e^{x-y}$

(f) $z_x = \ln y$ ; $z_y = \dfrac{x}{y}$

(g) $z_x = (-1)(x-2y)^{-2}.\ \dfrac{\partial}{\partial x}(x - 2y) = -(x - 2y)^{-2} = \dfrac{-1}{(x-2y)^2}$

$z_y = (-1)(x-2y)^{-2}.\ \dfrac{\partial}{\partial y}(x - 2y) = -(x - 2y)^{-2}(-2) = \dfrac{2}{(x-2y)^2}$

E 21) (a) $\dfrac{\partial z}{\partial x} = -\dfrac{x}{z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{y}{z}$

(b) $\dfrac{\partial z}{\partial x} = -\dfrac{2x+3y+3z}{3x + 2z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{4y - 3x}{3x + 2z}$

E 22) (a) $\dfrac{\partial z}{\partial x} = 5x^4y^4$, $\dfrac{\partial z}{\partial y} = 4x^5y^3$, $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2$

Therefore, $\dfrac{dz}{dt} = 22t^{21}$

(b) $\dfrac{dz}{dt} = y^2e^{xy^2}(\cos t - t\sin t) + 2xye^{xy^2}(\sin t + t\cos t)$

(c) $\dfrac{dz}{dt} = 4\sin 2t + 3\cos 2t$

(d) $\frac{dz}{dt} = \frac{-2(e^{-2t} - e^{2t})}{e^{-2t} + e^{2t}}$

E 23) (a) $\frac{\partial u}{\partial r} = (2x - y)\cos\theta + (2y - x)\sin\theta$

$\frac{\partial u}{\partial \theta} = (2x - y)(-r\sin\theta) + (2y - x)\, r\cos\theta$

(b) $\frac{\partial u}{\partial r} = 4x + 2y$ ; $\frac{\partial u}{\partial \theta} = -6x - 16y$

E 24) (a) $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (3x^2 e^{x/y} + \frac{x^3}{y}e^{x/y})\, dx + x^4 e^{x/y}\, dy$

(b) $df = (3x^2y - 4y^2)\, dx + (x^3 - 8xy + 24y^2)\, dy$

E 25) (a) $\frac{\partial z}{\partial x} = 2x + 5y$, $\frac{\partial z}{\partial y} = 5x + 2y$

$\frac{\partial^2 z}{\partial x^2} = 2$; $\frac{\partial^2 z}{\partial y^2} = 2$;

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 5 = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x}$$

(b) $f_{xx} = 0$, $f_{yy} = xa^2e^{ay}$

$f_{xy} = a\, e^{ay}$ and $f_{yx} = 0$.

Note that $f_{xy} \neq f_{yx}$ in this case. Can you guess why?

E 26) $U_{xx} = \frac{1}{\sqrt{2}}\frac{\pi}{4}$, $U_{yy} = -\frac{1}{\sqrt{2}}\frac{\pi}{4}$, $U_{xy} = U_{yx} = 0$.

E 27) (a) $f_{xy} = f_{yx} = \frac{1}{y}$

(b) $f_{xy} = f_{yx} = mnax^{n-1}\, y^{m-1}$

E 28) (a) $U = \dfrac{x^2\left[x^2\left(1 - \frac{y^2}{x^2}\right)\right]^3}{\left[x^2\left(1 + \frac{y^2}{x^2}\right)\right]^3}$

$$= x^2\left[\frac{1 - \left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)^2}\right]^3 = x^2 f\left(\frac{y}{x}\right)$$

Hence, U is homogeneous function of degree 2.

(b) It is of the form $x^n f\left(\frac{y}{x}\right)$ hence homogeneous function of degree n.

E 29) $\frac{\partial f}{\partial x} = 2(ax + hy)$ ; $\frac{\partial f}{\partial y} = 2(hx + by)$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f.$

E 30) By definition, $\sin U = \frac{x^2 + y^2}{x + y} = f$ (say).

But $\frac{x^2 + y^2}{x + y}$ is homogeneous of first degree.

Thus, using Euler's theorem we get

$x\frac{\partial f}{\partial x} + y\frac{df}{\partial y} = f.$

Now, $\frac{\partial f}{\partial x} = \cos U\frac{\partial U}{\partial x}$, $\frac{\partial f}{\partial y} = \cos U\frac{\partial U}{\partial y}$

Therefore, $x\cos U\frac{\partial U}{\partial x} + y\cos U\frac{\delta U}{\delta y} = \sin U$

Divide by cos U throughout and get the result.

# UNIT 8 THE INTEGRAL

**Structure**



## 8.1 INTRODUCTION

So far we concentrated only on that part of calculus which is based on the operation of the derivative, namely, 'differential calculus'. The second major operation of the calculus is integral calculus. The name 'integral calculus' originated in the process of summation, and the word 'integrate' literally means 'find the sum of'. Historically, the subject arose in connection with the determination of areas of plane regions. But in the seventeenth century it was realised that integration can also be viewed as the inverse of differentiation. Integral calculus consists in developing methods for the determination of integrals of any given function.

The relationship between the derivative and the integral of a function is so important that mathematicians have labelled the theorem that describes this relationship as the **Fundamental Theorem of Integral Calculus.**

In this unit, we will introduce the notions of antiderivative, indefinite integral and the notion of definite integral as the limit of a sum. The Fundamental Theorem of Integral Calculus is also discussed in this unit.

### Objectives

After reading this unit, you should be able to:

- compute the antiderivative of a given function,
- use the properties of indefinite integrals to compute integrals of simple functions,
- compute the definite integral of a function as the limit of a sum,
- compute the definite integral of a function using the Fundamental Theorem of Integral Calculus.

## 8.2 ANTIDERIVATIVES

So far, we have been occupied with the 'derivative problem', that is, the problem of finding the derivative of a given function. Some of the important applications of the calculus lead to the inverse problem, namely, given the derivative of a function, is it possible to find the function? This process is called **antidifferentiation** and the result of antidifferentiation is called an **antiderivative**. The importance of the antiderivative results partly from the fact, that scientific laws often specify the rates of change of quantities. The quantities themselves are then found by antidifferentiation.

To get started, suppose we are given that $f'(x) = 5$. Can we find $f(x)$? It is easy to see that one such function f is given by $f(x) = 5x$, since the derivative of $5x$ is 5. Before making any definite decision, consider the functions

$5x + 3, 5x - 8, 5x + \sqrt{2}$