UNIT 6 DIFFERENTIAL CALCULUS

Structure

6.1 INTRODUCTION

The word **'Calculus'** is a Latin word, which means a 'pebble' or a 'small stone'. The word 'calculate' is also derived from the same Latin word 'calculus'. Calculus is primarily concerned with two basic operations called differentiation and integration. Isaac Newton (1642-1727), the English mathematician and scientist, and Gottfried Wilhelm Leibniz (1646-1716), the German mathematician, are considered the inventors of calculus. Newton in 1665 and Leibniz in 1675 made their inventions independently. But Leibniz in 1684 was the first to introduce notations and symbols which continue till today with only a few modifications.

Newton (1642-1727)

Newton and Leibniz were both interested in the geometrical aspects of calculus such as finding areas bounded by curves and finding tangent lines of curves. It is in this connection only, they invented rules and techniques of differentiation and integration. It was only with the introduction of the concept of the "function", a word first used by Leibniz, that the interest shifted from geometry to what we call analysis today. In Unit 2 of Block 1, you have already studied about various types of functions. The concept of "function" occupies a central place in calculus. Other important topics include the concepts of limit and continuity of a function, rules of finding derivatives of various trigonometric, algebraic, inverse trigonometric functions, various methods of integration, definite integral etc. We shall be studying some of these concepts in this unit.

Leibniz (1646-1716)

Objectives

After reading this unit you should be able to

- compute the limit of a function,
- define and give examples of continuous function,
- compute the derivatives of algebraic, trigonometric, inverse, exponential and logarithmic functions,
- describe the geometrical significance of the derivative.

6.2 LIMITS

From Unit 1, you know that f is a function if f(x) is uniquely defined for every x in the domain. For instance, if x = pulse rate, and f(x) = body temperature of one patient, then x and y measured several times, does not **define** a function (ref. Unit 1). On the other hand, if x = biological species, and f(x) = the number of chromosomes associated with each x, then this **defines** a function.

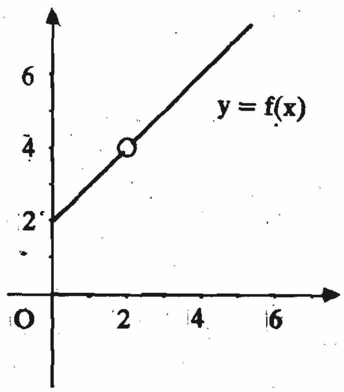


Fig. 1

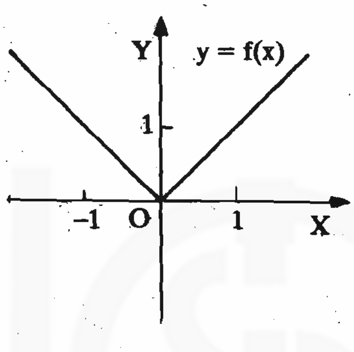
Firstly, we shall discuss the concept of the limit of a function at a point. Consider the function f: R → R such that $f(x) = \frac{x^2-4}{x-2}$. This function is defined for all $x \in \mathbf{R}$ (the set of real numbers) except at x = 2. At x = 2, $\frac{x^2-4}{x-2} = \frac{0}{0}$, is undefined.

Let us see what happens to f(x) as x gets closer and closer to 2 without actually taking the value 2 (see Fig. 1).

Table 1

x	.5	1.0	1.5	1.8	1.9	2.1	2.2	2.5	3.0	3.5
f(x)	2.5	3.0	3.5	3.8	3.9	4.1	4.2	4.5	5.0	5.5

If you look at the Table 1, you will notice that as the value of x increases gradually from .5 to 1.9 the corresponding value of f(x) increases from 2.5 to 3.9 which is very close to 4. All these values of x viz. .5, 1.0, 1.5, 1.8 and 1.9 etc. are less than 2. In this case we say that x is approaching 2 from the left of 2. Similarly, as x decreases from 3.5 to 2.1 the corresponding value of f(x) decreases from 5.5 to 4.1 again a value very close to 4. In this case we say that x is approaching 2 from the right of 2 since x in all these cases is greater than 2. Thus, x can approach 2 from either side of 2.



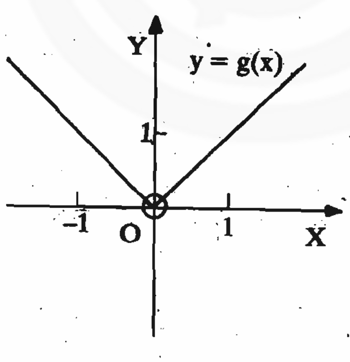
(a)

Also, as x gets nearer and nearer to 2 from either side of 2, f(x) gets closer and closer to 4. It means that if we take x **sufficiently** close to 2, then the numerical difference between the function $\frac{x^2-4}{x-2}$ and 4 can be made **small enough.**

This is expressed by saying that we say that the limit of $f(x) = \frac{x^2-4}{x-2}$ as x approaches 2 is 4. Formal definition of limit is now given below.

Definition : A function f is said to **tend toward** (or **approach** or **have a limit**) L as x approaches a number 'a' if for the values of x which are very close to a, the absolute value of the difference between f(x) and L is less than any preassigned positive number (however small it may be). This is expressed symbolically as $\lim_{x \to a} f(x) = L$ (read, "limit of f(x) as x tends to a is L").

Therefore, if $f(x) = \frac{x^2-4}{x-2}$, $\lim_{x \to 2} f(x) = 4$.



(b)

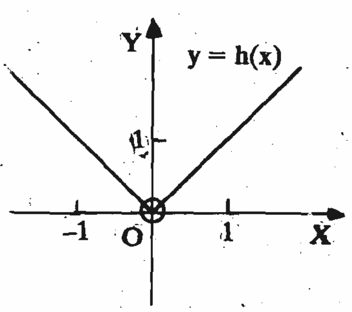
The limit L is a number associated with a function at a point (the point a). There are also other numbers associated with this function at the same point, for instance, f(a), the value of the function at a. While L and f(a) are frequently equal to each other, they need not always be so. In fact, either one or both may fail to exist; even if both exist, they may be unequal. For instance, if $f(x) = |x|$, then from the graph of $f(x) = |x|$ (Fig. 2(a)), we see that if x is close to zero, then so is f(x), infact x and f(x) are always the same distance from zero. Thus $f(x) \to 0$ as $x \to 0$ or $\lim_{x \to 0} f(x) = 0$.

Now, if

$$g(x) = \begin{cases} |x|, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Then $g(x) = f(x)$ for $x \neq 0$ (Fig. 2(b)). Thus, g(x) must have the same limit as f(x) as $x \to 0$, so $\lim_{x \to 0} g(x) = 0$.

This shows $\lim_{x \to 0} g(x) \neq g(0)$.



(c)

Fig. 2

Finally, consider the function $h(x) = |x|, x \neq 0$ which is undefined for x = 0 (Fig. 2(c)). In this case also, $\lim_{x \to 0} h(x) = 0$, despite the fact that no value is given for h(0). Since in all these cases the limit is zero as x approaches zero, it is clear that **the limit of a function at a point has nothing to do with the value that the function has at the point, or even with whether it has any value at all at that point.**

When $x \to a$ through values greater than a, x approaches 'a' from the right and this is expressed as $\lim_{x \to a^+} f(x) = L$. This is called the **right hand limit** of f at a. Similarly, if x is restricted to values less than a, we obtain the **left hand limit** of f at a, that is, $\lim_{x \to a^-} f(x) = L$.

The existence of the limit L implies that both the right hand and left hand limits exist and are equal, that is,

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

So far we have only considered limit at a finite point. However, it is also possible to consider limit of function at infinity, that is, as the variable becomes unbounded. If a variable x increases without bound, we say that x is approaching positive infinity and we write $x \to +\infty$. Similarly, if x decreases without bound, we say that x is approaching negative infinity, and we write $x \to -\infty$. If for instance,

$f(x) = \frac{1}{x}$, $x \neq 0$ then as $x \to \infty$ or $x \to -\infty$, f(x) takes on values closer and closer to zero. Thus, $\lim_{x \to +\infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0$.

Also, limit of a function **may or may not always exist** at a point. The function f itself may tend to become very large when x approaches a point a. For example, the function $f(x) = \frac{3}{2-x}$ tends to infinity as $x \to 2$ because $2-x \to 0$ as $x \to 2$.

Further, $\lim_{x \to 2^-} \frac{3}{2-x} = +\infty$, since as x approaches 2 from left, $(2-x)$ will always be positive tending to zero. Similarly, $\lim_{x \to 2^+} \frac{3}{2-x} = -\infty$. Hence, $\lim_{x \to 2} \frac{3}{2-x}$ **does not exist.** We will now state a theorem on limits which follows from the definition of a limit. We shall be assuming the theorem without actually proving it as the proof is beyond the scope of this course.

Theorem 1 : Let $f : \mathbf{R} \to \mathbf{R}$ and $\phi : \mathbf{R} \to \mathbf{R}$ be two functions.
Let $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} \phi(x) = B$. Then

a) If c is a constant; $\lim_{x \to a} c \cdot f(x) = c\,A$

b) $\lim_{x \to a} \{f(x) + \phi(x)\} = A + B$

c) $\lim_{x \to a} \{f(x) - \phi(x)\} = A - B$

d) $\lim_{x \to a} \{f(x)\phi(x)\} = A \cdot B$

e) $\lim_{x \to a} \{f(x)/\phi(x)\} = A/B$ provided $B \neq 0$.

These results are very important and will be used again and again for evaluating limits. We shall be illustrating this through various examples. But before that we would like to mention some **basic rules for evaluating limits.**

1) If the limit of f(x) has to be evaluated when $x \to a$, then substitute $x = h+a$ and evaluate the limit of $f(a+h)$ as $h \to 0$.
2) If the value of the function becomes 0/0 on putting $x = a$, then cancel any common factors between numerator and denominator and proceed as in (1).
3) If you have to evaluate limit $x \to \infty$, then put $x = 1/u$ and evaluate limit for $u \to 0$.

Let us now use these rules and Theorem 1 to solve the following examples:

Example 1 : Evaluate $\lim_{x \to 2} \frac{x^2+6}{x-1}$.

Solution : Substituting $x = 2$ in $\frac{x^2+6}{x-1}$ does not give us $\frac{0}{0}$.

So, we put $x = 2+h$, then

$$\frac{x^2+6}{x-1} = \frac{(2+h)^2+6}{2+h-1} = \frac{4+4h+h^2+6}{1+h} = \frac{10+4h+h^2}{1+h}$$

$$\lim_{x \to 2} \frac{x^2+6}{x-1} = \lim_{h \to 0} \frac{10+4h+h^2}{1+h} = 10.$$

Example 2 : Evaluate $\lim_{x\to 1} f(x)$ where $f(x) = \frac{x^2+4x-5}{x^2+x-2}$.

Solution : f(x) becomes 0/0 on putting x = 1.
Therefore, we write

$$\frac{x^2+4x-5}{x^2+x-2} = \frac{(x+5)(x-1)}{(x+2)(x-1)} = \frac{x+5}{x+2}$$

(This way we get rid of the troublesome factor (x−1) :) and so now

$$\lim_{x\to 1} \frac{x^2+4x-5}{x^2+x-2} = \lim_{x\to 1} \frac{x+5}{x+2} = \frac{1+5}{1+2} = \frac{6}{3} = 2.$$

Example 3 : Evaluate $\lim_{x\to c} x^n$, for each positive integer n.

Solution : For n = 1, $\lim_{x\to c} x^n = \lim_{x\to c} x = c$

For $n \geq 2$, let $f_1(x) = x,\ f_2(x) = x,\ \ldots\ldots,\ f_n(x) = x$

Then $\lim_{x\to c} x^n = \lim_{x\to c} \{ f_1(x)\cdot f_2(x) \ldots\ldots f_n(x) \}$

$= (\lim_{x\to c} x) \ldots\ldots (\lim_{x\to c} x)$ [using Theorem 1d]

$= c \ldots\ldots c = c^n$.

You may now try this exercise.

E1) Evaluate the following limits:

a) $\lim_{x\to 3} \frac{x-3}{x^2+x-12}$

b) $\lim_{x\to \infty} \frac{3x-1}{7x+6}$ (**Hint** : divide numerator and denominator by x then take limit)

c) $\lim_{x\to 3} \frac{x^2}{x-3}$

d) $\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$ (**Hint** : multiply the numerator and the denominator by $\sqrt{1+x} + \sqrt{1-x}$ and then take the limit)

e) If $f(x) = x^2$ find $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

Now after doing this exercise you must have got good practice about the evaluation of a limit of a function at a point. We now familiarise you with the concept of **continuity** and **continuous function.** The term "continuous function" was used by Euler and other eighteenth century mathematicians, but in a sense different from the one in use today. Bernhard Bolzano (1781-1848), a Bohemian priest and scholar gave essentially the modern definition of continuity and developed some of its consequences in a paper written in 1817. Meanwhile, Akugustin Louis Cauchy (1789-1857), a French mathematician, used the same definition a few years later and it was through him that the notion of continuity became well known.

6.3 CONTINUITY

Before giving the formal definition of continuity, we would like you to understand the concept of continuity of a function through some examples.

Example 4 : Discuss the continuity of the function f : **R** → **R** defined by

a) $f(x) = 2x + 1$

b) $f(x) = x^2$

c) $f(x) = 1/x$

d) $f(x) = x^2$ when $x \neq 1$
$\quad = 2$ when $x = 1$

Solution : The functions (a) and (b) are well defined for all $x \in \mathbf{R}$. Their graphs are unbroken and there are no jumps in them. (See Fig. 3)

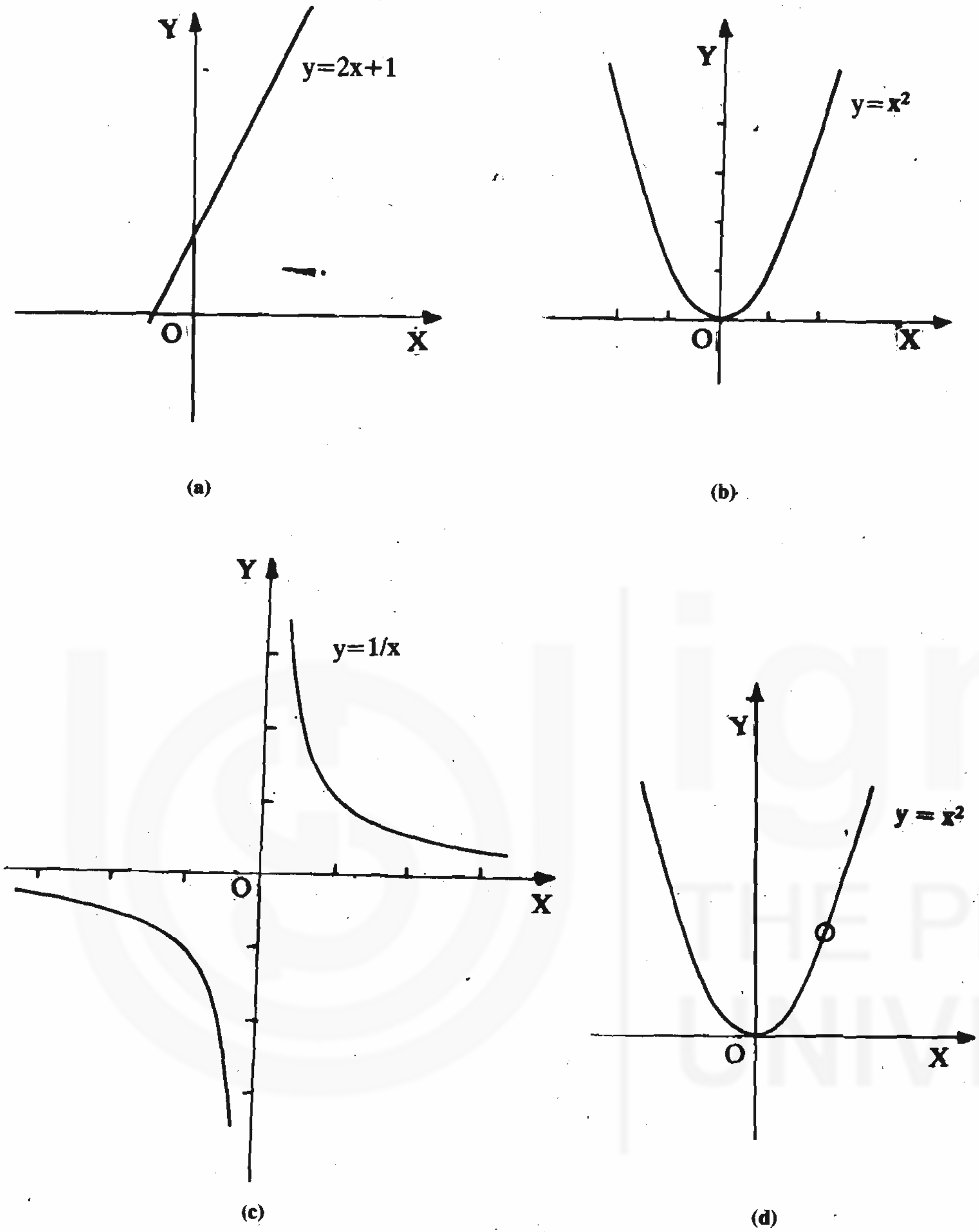


Fig. 3

While drawing the graphs of these functions the pencil never leaves the paper. Function (c) is not defined at $x = 0$. In fact even the $\lim_{x \to 0} f(x)$ does not exist because $f(x) = \infty$ for $x \to 0^+$ and $f(x) = -\infty$ as $x \to 0^-$.

The function (d) is the same as function (b) except at $x = 1$. At $x = 1$ the given **value** of the function (d) is 2, but the **limit** of the function is 1. Thus, at $x = 1$, the limit of the function is not equal to the value of the function. Therefore, there is a **jump** in the function at $x = 1$.

In this example, the two functions (a) and (b) are said to be continuous functions for all $x \in \mathbf{R}$. The function (c) is not continuous at $x = 0$, whereas the function (d) is not continuous at $x = 1$.

Let us now consider another example to see what we are trying to say about continuity.

Example 5 : Discuss the continuity of the function f where

$$\begin{aligned} f(x) &= 2 \text{ for } 0 < x \leq 1 \\ &= 3 \text{ for } 1 < x \leq 2 \\ &= 4 \text{ for } 2 < x \leq 3 \end{aligned}$$

Solution : This function f is known as a step function; and it has jumps at $x = 1, 2, 3$ (see Fig. 4).

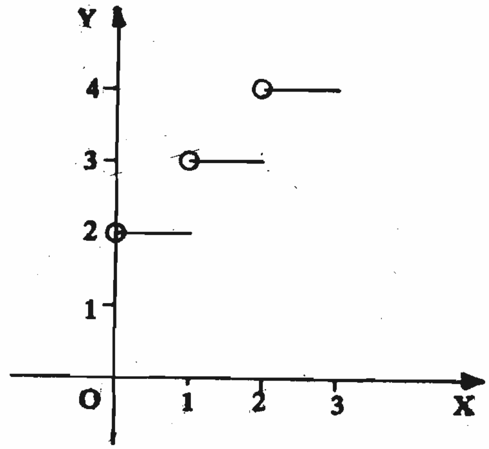


Fig. 4

It is clear from Fig. 4 that at $x = 1, 2, 3$ the left limit and the right limit are not equal and we say that the function is not continuous at points $x = 1, 2$ and 3 in the interval $(0 < x \leq 3)$. Conclusions drawn from the above two examples can be summed up as follows.

The function f is **not continuous** or in other words is **discontinuous** at $x = a$ if any of the following three holds:

i) the function is not defined at $x = a$.

ii) the limit of the function f does not exist at $x = a$.

iii) the limit of f at $x = a$ exists but is not equal to the value of f at a that is, f(a).

In drawing a graph of a continuous function, the pencil need never leave paper, while there is generally a jump in graph of a discontinuous function. This is of course merely a property and not a definition of continuity or discontinuity. Thus, we now give the formal definition of the continuity of a function at $x = a$, a point in its domain.

Definition : Suppose the domain of a function f includes an open interval containing the point a, then f is continuous at a, if

i) the limit as $x \to a$ of f(x) exists.

ii) $\lim_{x\to a} f(x) = f(a)$, the value of f at the point $x = a$.

Also, a function is continuous in the interval]a,b[if it is continuous at every point of the interval]a,b[.

Further, if a function is discontinuous at a point $x = a$, in its domain, but has a **finite limit** L at $x = a$, then we can make the function continuous at $x = a$, by redefining it and **assigning** it the value L at $x = a$. Let us now illustrate this fact through the following example.

Example 6 : Discuss the continuity of the function f defined by

$$f(x) = \frac{x^2-9}{x-3}, \text{ at } x = 3.$$

Solution : Function f is of the form 0/0 at $x = 3$.
Therefore, it is not defined at $x = 3$. But the limit,

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3} (x+3) = 6.$$

Rational functions are continuous at all points where denominator is non-zero.

Thus, f has a finite limit at $x = 3$.
Therefore, if we assign the value 6 to f(x) at $x = 3$, then the function f defined by

$$f(x) = \frac{x^2-9}{x-3}, \text{ for } x \neq 3$$

$$= 6 \text{ for } x = 3.$$

is continuous at $x = 3$.

You can now try the following exercises.

E2) For what values of x are the following functions not continuous?

a) $\frac{3x+1}{x^2-9}$ b) $\frac{x+3}{x^2+2}$ c) $\frac{3}{x+2}$

E3) What value assigned to f(x) at x = 8, will make the function f defined by

$f(x) = \frac{x^2-64}{x+2}$ continuous?

E4) Is the function f defined by $f(x) = \frac{3}{x+2}$ continuous at x = −2? If not, can you redefine it to make it continuous at x = −2?

Now you are well familiar with the meaning of limit of a function at a point. The process of limiting can be applied to a function f purely as a mathematical operation, without regard to its possible interpretation in terms of physics or geometry. This mathematical process about which we shall be studying now is known as **differentiation.**

6.4 DIFFERENTIATION

The operation of differentiation when applied to any function yields a result which is called its derivative.

6.4.1 Derivative of a Function at a Point

Given a function f, the main object of differential calculus is to find, how the function changes, when a small change is made in the variable x. In particular, in differential calculus one is interested in **the rate** at which f(x) changes with respect to x. In this section, we shall derive the formula which gives a mathematical expression to this notion. Consider the function y = f(x), $x \in \mathbf{R}$. The small change in x, whether positive or negative is called an **increment** in x and is denoted by δx (pronounced delta x). Corresponding to this change in x, there is a change in y, we denote it by δy. Thus, we have

δ is the Greek letter "delta" stands for difference. The combination "δx" is not the product of δ and x but rather a single quantity.

$y+\delta y = f(x+\delta x)$

Therefore, $\delta y = f(x+\delta x) - y = f(x+\delta x) - f(x)$.

Thus, the average rate of change $\frac{\delta y}{\delta x}$, is given by

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

If this quotient tends to a limit as $\delta x \to 0$, then this limit is called the **derivative** of the function y with respect to x and is denoted by $\frac{dy}{dx}$

In the above expression if we replace δx by h, then we can write

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

We thus have the following definition.

Definition : A function f whose domain includes an open interval containing the point x_0 is said to be derivable at $x = x_0$, if

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \text{ exists.}$$

The limit is denoted by $f'(X_0)$ and is called the derivative (or differential co-efficient) of the function f at $x = x_0$.

The operation of finding the derivative is called differentiation.

Remember that the limit must be the same whether h tends to zero through positive or through negative values.

We now make use of the above definition in the following examples.

Example 7 : Find the derivatives of

i) x^2, ii) $\sqrt{x}$, iii) $1/x$, $x \in \mathbf{R}$

Solution : i) $y = f(x) = x^2$.

Thus, $f(x+\delta x) = (x+\delta x)^2$ and $f(x+\delta x) - f(x) = (x+\delta x)^2 - x^2$

$$\text{Therefore, } \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^2+2x\delta x+(\delta x)^2 - x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} (2x+\delta x) = 2x$$

Hence, $\frac{d}{dx}(x^2) = 2x$.

ii) $f(x) = \sqrt{x}$. Here, $f(x+\delta x) = \sqrt{x+\delta x}$

$$\text{Therefore, } \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sqrt{x+\delta x} - \sqrt{x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sqrt{x+\delta x} - \sqrt{x}}{\delta x} \times \frac{\sqrt{x+\delta x} + \sqrt{x}}{\sqrt{x+\delta x} + \sqrt{x}}$$

$$= \lim_{\delta x \to 0} \frac{x+\delta x - x}{(\sqrt{x+\delta x} + \sqrt{x})\,\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\sqrt{x+\delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Thus, $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$.

iii) $y = \frac{1}{x} = x^{-1} = f(x)$ and $f(x+\delta x) = \frac{1}{x+\delta x}$

$$\text{In this case, } \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{1}{x+\delta x} - \frac{1}{x}\right]$$

$$= \lim_{\delta x \to 0} \left\{\frac{x - (x+\delta x)}{\delta x \cdot x(x+\delta x)}\right\}$$

$$= \lim_{\delta x \to 0} \frac{-1}{x(x+\delta x)} = -\frac{1}{x^2}$$

Therefore $\frac{d}{dx}(x^{-1}) = \frac{-1}{x^2}$.

Remark : In Example 7, you will notice that the derivative obtained in (i), (ii) and (iii) are $2x^{2-1}$, $1/2\,(x^{1/2-1})$ and $(-1)x^{-1-1}$ respectively. They all follow the same pattern namely, $\frac{d}{dx}(x^n) = n\,x^{n-1}$ for different values of n.

Let us now see if this formula is true in general.

6.4.2 Derivative of x^n

Let $y = f(x) = x^n$, $x \in \mathbf{R}$. Then,

$y+\delta y = f(x+\delta x) = (x+\delta x)^n$. Thus,

$\delta y = f(x+\delta x) - f(x) = (x+\delta x)^n - x^n$

$$= x^n + nx^{n-1}\cdot(\delta x) + \frac{n(n-1)}{2!}x^{n-2}\cdot(\delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}\cdot(\delta x)^3 + \ldots\ldots + (\delta x)^n - x^n$$

using binomial expansion (Refer Block 1)
Therefore,

$$\frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{2!} x^{n-2} \cdot \delta x + \frac{n(n-1)(n-2)}{3!} (\delta x)^2 + \ldots\ldots\ldots\ldots$$

.... + terms containing higher powers of δx.

Hence,

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = nx^{n-1}$$

and this shows that the formula is true in general for all real n. You will now notice that this formula is very convenient for finding the derivatives of various functions.

For examples :

$$\frac{d}{dx}(x^8) = 8x^{8-1} = 8x^7 \text{ (putting n=8)}$$

$$\frac{d}{dx}(x^{-5}) = (-5)x^{-5-1} = -5x^{-6} \text{ (for n = -5) and so on.}$$

You can now try the following exercises.

E5) Write down the derivatives of the following :

a) x^3 b) x^{-1} c) $x^{3/2}$

E6) Compute the following :

a) If u = v, du/dv = ?
b) If $s = 4/r^2$, ds/dr = ?
c) If $z = x^{1/3}$, dz/dx = ?
d) If $x = t^{2/5}$, dx/dt = ?
e) If $R = u^{12}$, dR/du = ?
f) If $y = x^{-9}$, dy/dx = ?

6.4.3 Geometric Significance of the Derivative

You know that the function y = f(x) represents a curve in the xy-plane. Let the graph of this curve be represented in (OX, OY) plane by C. (see Fig. 5).

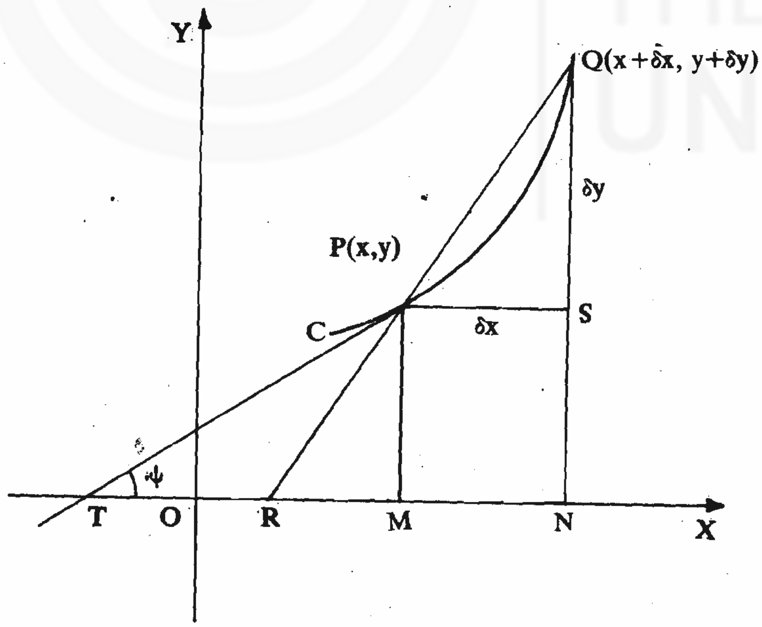


Fig. 5

Take P(x,y) and Q(x+δx, y+δx) as two neighbouring points on this curve C. Draw PM and QN perpendicular to the x-axis and PS perpendicular to QN. Then PS = δx, QS = δy and δy/δx = QS/PS. Join QP and produce it to meet the x-axis at R. Let /PRX = θ. Now, if Q moves along the curve towards P, the chord QP will move through P and in the limiting case when Q coincides with P, the chord QP will become the tangent at P. Let PT be this tangent.

Thus, when $Q \to P$, $\angle PRX$ will become $\angle PTX = \psi$, which the tangent at P makes with the axis OX.

Then $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \tan \psi$ = the slope of the tangent PT = the slope of the curve C at P.

Therefore, the derivative of the function f(x) with respect to x at a point P on the curve y = f(x) gives the slope of the curve at that point. For instance, the slope of the curve $y = x^2$ at a point (1,1) on it is $\left.\frac{dy}{dx}\right|_{(1,1)} = 2(x)_{x=1} = 2$. Thus, $\tan \psi = 2$. Also, the slope at (0,0) is 0 and at (2,4) is 4.

E7) Find the points on the graph of $y = x^3/3 - x^2 - 2x - 3$ where the tangent is parallel to the line $x - y - 6 = 0$.

Hint : Parallel lines have the same slope (ref. Unit 4).

6.4.4 Rules for Differentiation

Let us now consider various rules of finding derivatives of different types of functions. These rules follow immediately from the definition by making use of Theorem 1. But here we shall be just stating them without giving their proofs.

Rule 1 : The derivative of a constant is zero. This is intuitively obvious if we think of the derivative as the rate of change of y with respect to x. Since a constant does not change, you will at once say that the rate is zero.

Thus if y = f(x) = k, k a constant, then $\frac{dy}{dx} = \frac{d(k)}{dx} = 0$.

Rule 2 : The derivative of the product of a constant and a function is equal to the product of that constant and the derivative of the function. That is,

$\frac{d}{dx}\{cf(x)\} = cf'(x)$, where c is a constant.

Example 8 : For $y = 7x^2$ find $\frac{dy}{dx}$.

Solution : Here, we take $f(x) = x^2$. Then,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{7f(x+h) - 7f(x)}{h} = \lim_{h \to 0} \frac{7(x+h)^2 - 7x^2}{h}$$

$$= \lim_{h \to 0} \frac{14 \times h + 7h^2}{h} = 14x = 7 \cdot 2x = 7f'(x)$$

Rule 3 : The derivative of sum or difference of two functions is equal to the sum or difference of their respective derivatives. That is, if u and v are two functions of x then,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}.$$

This result is also valid for the sum or difference of more than two functions.

Note : Rule 2 is also applicable to the product of a constant with sum or difference of two or more functions, that is, $\frac{d}{dx}\{c[f(x) \pm g(x) \pm h(x)]\} = cf'(x) \pm cg'(x) \pm ch'(x)$.

Rules 1-3 will become more clear from the following worked-out example.

Example 9 : If $f(x) = 3x^2 + 5\sqrt{x} - 2x$, find $f'(x)$.

Solution : $f'(x) = \frac{d}{dx}[f(x)] = \frac{d}{dx}[3x^2 + 5\sqrt{x} - 2x]$

$$= \frac{d}{dx}(3x^2) + \frac{d}{dx}(5\sqrt{x}) - \frac{d}{dx}(2x)$$

$$= 3\frac{d}{dx}(x^2) + 5\frac{d}{dx}(\sqrt{x}) - 2\frac{d}{dx}(x)$$

$$= 3.2x + 5\ .\ 1/2.x^{-1/2} - 2.1$$

$$= 6x + 5/2(x^{-1/2}) - 2.$$

You can now try the following exercises :

E8) Find $f'(x)$ where

a) $f(x) = 3x^5 + 2\sqrt{x}$ b) $f(x) = \frac{x^5 + 4x^3 - 11}{x^4}$

c) $f(x) = \frac{x^2+x+1}{\sqrt{x}}$

E9) a) If $u = v^{3/5} - 6v + 8$. find du/dv.

b) If $r = 3s^6 - 5s^4 + 6s^2 - 1$, find dr/ds.

Let us now consider two more rules for differentiation.

Rule 4 : The derivative of the product of two functions = first function × derivative of the second function + second function × the derivative of the first.

In other words, if u and v are any two functions of x then,

$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$, this is known as the **product rule** for differentiation.

Example 10 : Differentiate with respect to x the function $y = (3x+7)(5-2x)$.

Solution : To get $\frac{dy}{dx}$, let $u = 3x+7$ and $v = 5-2x$. Then $y = u \cdot v$

Now, $\frac{du}{dx} = 3$ and $\frac{dv}{dx} = -2$. Using rule 4,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (3x+7)(-2) + (5-2x)(3) = 1-12x.$$

E10) Differentiate the following with respect to x :

a) $(x^2+7)(3-x)$ b) $(x^3+3x)(x^2+2)$ c) $(x^2-1)(x^3-x+2)$

Rule 5 : The derivative of the quotient of two functions = [denominator × derivative of the numerator − numerator × derivative of the denominator] ÷ square of the denominator.

That is, if u and v are functions of x then,

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du/dx - u\,dv/dx}{v^2}$, this is called the **quotient rule for differentiation.**

For example, if $y = \frac{2x^2+3}{8-3x^2}$, then for $u = 2x^2 + 3$ and $v = 8-3x^2$,

$\frac{du}{dx} = 4x;\ \frac{dv}{dx} = -6x$ and using Rule 5,

$$\frac{dy}{dx} = \frac{(8-3x^2)(4x) - (2x^2+3)(-6x)}{(8-3x^2)^2} = \frac{32x-12x^3+12x^3+18x}{(8-3x^2)^2}$$

$$= \frac{50x}{(8-3x^2)^2}.$$

You can now do this exercise very easily.

E11) Find the derivative of the following functions :

a) $\frac{2x-5}{3x+4}$ b) $\frac{3x^2}{2x^2+3}$ c) $\frac{x^{1/2}-x^{-1/2}}{x^{1/2}+x^{-1/2}}$ d) $\frac{x^3+3x}{x^2+2}$

We will now give differentiation formulae for the six trigonometric functions using these rules for differentiation as well as the definition of the derivative. Remember that in all discussions involving the trigonometric functions radian measure is used.

6.5 DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

Trigonometric functions are of particular significance in life sciences because of their periodicity. Biological rhythms such as heart beats or pulse beats are periodic. Also, leaf arrangements, spirals, bird orientations are best represented through polar coordinates involving trigonometric functions.

In this section, we shall derive the derivatives of trigonometric functions and in the process use an important result on limit namely $\lim_{x\to 0} \frac{\sin x}{x} = 1$. We shall not be giving you the proof of this result. Apart from this result, we shall also be frequently using, various trigonometric identities such as,

a) $\cos^2 x + \sin^2 x = 1$ b) $1 + \cot^2 x = \operatorname{cosec}^2 x$

c) $1 + \tan^2 x = \sec^2 x$ d) $\sin 2x = 2\sin x \cos x$

Derivative of sin x

Write y = sin x. Let δy be the change in y corresponding to change δx in x. Then, $y+\delta y = \sin(x+\delta x)$ and

$$\delta y = \sin(x+\delta x) - \sin x$$

$$= 2\cos(x+\delta x/2) \sin \delta x/2 \text{ (Refer Block 1)}$$

$$\frac{\delta y}{\delta x} = \frac{2\cos(x+\delta x/2)\sin \delta x/2}{\delta x}$$

$$= \cos(x+\delta x/2) \cdot \frac{\sin \delta x/2}{\delta x/2}$$

Thus, $\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \cos x$, since $\frac{\sin \delta x/2}{\delta x/2} \to 1$ as $\delta x \to 0$

or, $\frac{d}{dx}(\sin x) = \cos x.$

Exactly on the similar lines we get

$$\frac{d}{dx}(\cos x) = -\sin x.$$

$\frac{d}{dx}(\tan x)$ can be obtained using quotient rule. Now,

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\, d/dx(\sin x) - \sin x\, d/dx(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= 1/\cos^2 x = \sec^2 x.$$

Similarly using quotient formula, we have

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}(1/\cos x) = \frac{\cos x \cdot 0 - 1\cdot(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = d/dx(\cos x/\sin x) = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2x - \cos^2x}{\sin^2x} = -1 - \cot^2x = -\text{cosec}^2x$$

$$\frac{d}{dx}(\text{cosec } x) = \frac{d}{dx}(1/\sin x) = \frac{\sin x \cdot 0 - 1 \cdot \cos x}{\sin^2x}$$

$$= \frac{-\cos x}{\sin^2x} = \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\cot x \text{ cosec } x$$

Let us now sum up these results in the form of a table.

Table 2

f(x)	$\frac{df}{dx}$
sin x	cos x
cos x	− sin x
tan x	$\sec^2x$
sec x	sec x tan x
cot x	$-\text{cosec}^2x$
cosec x	− cot x cosec x

The usefulness of these formulae is well illustrated in the following example and exercise.

Example 11: Find $\frac{dy}{dx}$, where

i) $y = \sin x \cdot \sec x$ ii) $y = x^5 \cos x$ iii) $y = \tan x + \cot x$

Solution:

i) $\frac{d}{dx}(\sin x \sec x) = \sin x \frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(\sin x)$

$= \sin x \sec x \cdot \tan x + \sec x \cos x$

$= \sin x (1/\cos x) \tan x + (1/\cos x) \cos x$

$= \tan^2x + 1 = \sec^2x.$

ii) $\frac{d}{dx}(x^5 \cos x) = x^5 \cdot (-\sin x) + \cos x \cdot 5x^4$

$= -x^5 \sin x + 5x^4 \cos x.$

iii) $\frac{d}{dx}(\tan x + \cot x) = \frac{d}{dx}(\tan x) + \frac{d}{dx}(\cot x)$

$= \sec^2x - \text{cosec}^2x.$

And now an exercise for you.

E12) Find the derivative of the following functions :

a) $x^3 \sin x$ b) $\sin x \cos x$ c) $(3x+2)/\cos x$

d) $x \text{ cosec } x$ e) $\frac{\tan x - \cot x}{\tan x + \cot x}$

For two given functions u and v we have already stated in Sec. 6.4 how to express the derivative of their sum, difference, product or quotient in terms of u, v and their derivatives. We now consider the method of finding derivative of composite function.

6.6 COMPOSITE FUNCTIONS — CHAIN RULE

In Unit 1, Sec. 1.8 of Block 1, you have already learnt about composite function. But, before finding the rule for differentiating composite functions let us once again recall its definition.

Definition : If y is a function of u where u itself is a function of x, then the result of substituting u into y is called **composite function** of x or a **function of a function.** Thus, for $y = f(u)$ where $u = \phi(x)$, we use the notation $f[\phi(x)]$ to denote y, a composite function of x. In order to be able to evaluate $f[\phi(x)]$ it is necessary that the value of $\phi(x)$ be contained in the domain of f.

We now obtain the derivative of this composite function. Let δy and δu be the increments in y and u respectively, corresponding to a change δx in x.

Then, we can write

$\frac{\delta y}{\delta x} = \left(\frac{\delta y}{\delta u}\right) \cdot \left(\frac{\delta u}{\delta x}\right)$ (since each of $\frac{\delta y}{\delta u}$ and $\frac{\delta u}{\delta x}$ is a quotient).

Taking limits, as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \cdot \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

Since $\delta x \to 0$ will mean $\delta u \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \cdot \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

In other words, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

This is an important and extensively used technique in differentiation.

Let us now solve some examples using this technique.

Example 12 : If $y = (ax+b)^n$, find $\frac{dy}{dx}$.

Solution : Put $ax+b = u$. Then, $y = u^n$ and

$\frac{dy}{du} = nu^{n-1}$, but $\frac{du}{dx} = a$

Therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= nu^{n-1} \cdot a$

$= na(ax+b)^{n-1}$

Working rule : Given a composite function $f[\phi(x)]$, first differentiate the function f with respect to $\phi(x)$ and then multiply the result with the derivative of $\phi(x)$ with respect to x.

Look at another example.

Example 13 : Find $\frac{d}{dx}$ (sin δx).

Solution : Let $y = \sin 5x$ and $u = 5x$. Then, $y = \sin u$.

Using chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \cos u \cdot 5$

$= 5 \cos 5x.$

Note that in this example to obtain the derivative of sin 5x we have first differentiated sin 5x to get cos 5x and then multiplied it by 5 that is, the derivative of 5x to get 5 cos 5x. This point is further illustrated in the following example.

Example 14 : Find i) $\frac{d}{dx}(x^3 + 3x^2 - 5x + 1)^3$,   ii) $\frac{d}{d\theta}(\sin^3\theta)$

Solution : i) $\frac{d}{dx}(x^3+3x^2-5x+1)^3 = 3(x^3+3x^2-5x+1)^{3-1} \frac{d}{dx}(x^3+3x^2-5x+1)$

$= 3(x^3+3x^2-5x+1)^2 \cdot (3x^2+6x-5)$

ii) $\frac{d}{d\theta}(\sin^3\theta) = \frac{d}{d\theta}(\sin\theta)^3 = 3(\sin\theta)^2 \cdot \frac{d}{d\theta}(\sin\theta)$

$= 3\sin^2\theta \cos\theta.$

You can now try the following exercise.

E13) Find the derivatives of the following functions.

a) $\sin 3x$ b) $\cos 5x$ c) $x^2 \tan x$

d) $\sqrt{x^2+1}$ e) $\frac{(2x-1)^3}{(x+1)}$ f) $\cos\sqrt{x}$

g) $\frac{\sin^2 x}{1+\cos x}$

Let us now see how the chain rule can be used to obtain the derivatives of some more functions.

6.6.1 Differentiation of Exponential Functions

In Unit 2, Sec. 2.3.1 of Block 1, you have already familiarised yourself with exponential functions. An exponential function is defined by $y = a^x$ where $x \in \mathbf{R}$ and a is a positive constant. In particular, a = e where e is defined as $\lim_{n\to\infty} (1+1/n)^n \approx$ 2.7182818 provides the natural base for exponential functions. Whenever a quantity is changing at a rate proportional to the magnitude of the quantity, we can look for a relation based on e^x because e^x remains unchanged after differentiation.

(i.e., $\frac{d}{dx} e^x = e^x$).

In Chemistry, the reaction rates, chemical equilibria phenomenon of solubility, vapour pressure etc., depend on the law of exponential growth. Similarly, in Biology, the weight increase of a plant, or the doubling of the growth of number of cells also follow this law.

While calculating the derivative of e^x we shall be making use of the fact that

$\lim_{h\to 0} \frac{e^h-1}{h} = 1$, without giving its proof.

Now let $y = e^x$ and let δy be the change in y corresponding to a change δx in x, then

$$\delta y + y = e^{x+\delta x}, \text{ therefore } \delta y = e^{x+\delta x} - e^x$$
$$= e^x(e^{\delta x} - 1)$$

Thus, $\frac{\delta y}{\delta x} = \frac{e^x(e^{\delta x}-1)}{\delta x}$

$$\text{or } \frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0} e^x \frac{e^{\delta x}-1}{\delta x}$$
$$= e^x \lim_{\delta x\to 0} \frac{e^{\delta x}-1}{\delta x} = e^x \cdot 1 = e^x$$

Hence, $\frac{d}{dx}(e^x) = e^x$.

The derivative $\frac{d}{dx}(e^{ax})$ where, a is any constant can now be easily obtained, by writing,

$$\frac{d}{dx}(e^{ax}) = e^{ax} \cdot \frac{d}{dx}(ax) = ae^{ax}.$$

For example, we have

$$\frac{d}{dx}(e^{3x}) = e^{3x}\frac{d}{dx}(3x) = e^{3x} \cdot 3 = 3e^{3x}$$

$$\frac{d}{dr}(e^{2r+3}) = e^{2r+3}\frac{d}{dr}(2r+3) = 2 \cdot e^{2r+3}$$

$$\frac{d}{ds}(e^{2/s}) = e^{2/s}\frac{d}{ds}(2/s) = e^{2/s}(-2/s^2) = -2/s^2 (e^{2/s})$$

$$\frac{d}{dx}(e^{\cos x}) = e^{\cos x}\frac{d}{dx}(\cos x) = -\sin x \cdot e^{\cos x}$$

Note : We apply the same method to find the derivative, when a is not a constant, but is any function of x.

Differentiation of a^x : Put $a = e^u$ so that $a^x = e^{ux}$ and $u = \ln a$.

Then, $\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{ux}) = e^{ux} \cdot u = a^x \cdot \ln a$

Thus, $\frac{d}{dx}(a^x) = a^x \cdot \ln a$

Notation ln a represents $\log_e a$

The differential of $\frac{d}{dx}(a^{f(x)})$ for any general form of the function f(x) can also be obtained as above by using chain rule in the form

$$\frac{d}{dx}(a^{f(x)}) = a^{f(x)} \ln a \cdot \frac{d}{dx}(f(x)) = a^{f(x)} \ln a \cdot f'(x).$$

Using this formula, we can write

$$\frac{d}{dx}(2^x) = 2^x \ln 2$$

$$\frac{d}{dx}(4^{x^3}) = 4^{x^3} \ln 4 \cdot \frac{d}{dx}(x^3) = 4^{x^3} \ln 4 \cdot 3x^2$$

$$\frac{d}{dx}(5^{\sin x}) = 5^{\sin x} \ln 5 \cdot \frac{d}{dx}(\sin x) = 5^{\sin x} \ln 5 \cdot \cos x$$

You can now try this exercise.

E14) Find the derivatives of the following functions.

a) $\frac{d}{dx}(3^{-5x})$ b) $\frac{d}{d\theta}(e^{\sin \theta})$ c) $\frac{d}{dr}(e^{-2r})$

d) $\frac{d}{ds}(e^{x^2})$ e) $\frac{d}{dx}(e^{x\cos x})$

We now calculate the derivative of natural logarithmic function, that is, logarithm to the base e.

6.6.2 Differentiation of Logarithmic Functions

The logarithmic function, as you already know from Unit 2, Sec. 2.3.2, is defined as the inverse of the exponential function. In order to calculate its derivative let $y = \ln x$ and δy be the increment in y corresponding to change δx in x.

Now, $y + \delta y = \ln(x + \delta x)$ and so

$$\delta y = \ln(x+\delta x) - \ln x = \ln \frac{x+\delta x}{x} = \ln(1+\delta x/x)$$

$$\text{Therefore, } \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta x}{\delta x} = \lim_{\delta x \to 0} \frac{1}{\delta x} \ln(1+\delta x/x).$$

Note that ln is a continuous function, therefore

$$\lim_{\delta x \to 0} \ln\left(1+\frac{\delta x}{x}\right) = \ln \lim_{\delta x \to 0}\left(1+\frac{\delta x}{x}\right)$$

On multiplying and dividing by x, the above equality becomes,

$$\frac{dy}{dx} = 1/x \lim_{\delta x \to 0} \frac{x}{\delta x} \ln(1+\delta x/x)$$

$$= 1/x \lim_{\delta x \to 0} \ln(1+\delta x/x)^{x/\delta x}$$

Since $\lim_{\delta x \to 0} (1+\delta x/x)^{x/\delta x} = e$, we get $\frac{dy}{dx} = 1/x \ln e = 1/x$

(because $\ln e = 1$)

Therefore, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

By making use of the chain rule and above formula we have, for any arbitrary function f(x),

$$\frac{d}{dx}[\ln f(x)] = 1/f(x) \cdot \frac{d}{dx}[f(x)] = [1/f(x)] \cdot f'(x)$$

For example,

$$\frac{d}{du}(\ln u) = 1/u$$

$$\frac{d}{d\theta}[\ln(\sin \theta)] = 1/\sin\theta \cdot \frac{d}{d\theta}(\sin\theta) = \cos\theta/\sin\theta = \cot\theta$$

$$\frac{d}{ds}[\ln(1+1/s)] = \frac{1}{1+1/s}\;\frac{d}{ds}(1+1/s)$$
$$= \frac{1}{1+1/s}\cdot(-1/s^2) = s/1+s\cdot(-1/s^2)$$
$$= -1/s(1+s).$$

Using the above formula, the derivative of the product of two functions, one of them being the logarithmic function can also be obtained very easily as illustrated in the following.

Example 15 : Find $\frac{d}{dr}\ln[r\cdot\ln(2r)]$

Solution : $\frac{d}{dr}\ln[r\cdot\ln(2r)] = \frac{1}{r\ln 2r}\cdot\frac{d}{dr}[r\cdot\ln(2r)]$

$= \frac{1}{r\ln(2r)}[r\cdot\frac{d}{dr}(\ln 2r) + \ln 2r\cdot\frac{d}{dr}(r)]$ (using product rule)

$= \frac{1}{r\ln(2r)}[r\cdot 1/2r\frac{d}{dr}(2r) + \ln 2r\cdot 1]$

$= \frac{1}{r\ln(2r)}[1/2\cdot 2 + \ln 2r\cdot 1]$

$= \frac{1}{r\ln(2r)}[1 + \ln 2r)$

How about doing an exercise now?

E15) Find the following derivatives.

a) $d/du\,[\ln(u^2-3)]$ b) $d/dt\,[\ln(t\cdot e^t)]$ c) $d/dx\,[\ln\sqrt{x}]$

d) $d/d\theta\,[\ln\sin\sqrt{\theta}]$ e) $d/dx\,(x^2/\ln x)$ f) $d/dr\,[3r\ln(r^2)]$

6.6.3 Differentiation of Functions Defined by Means of a Parameter

Suppose that x and y are given as functions of another variable t, by equations of the form $x = f(t)$, $y = \phi(t)$. Then these equations taken together are said to be **parametric equations** with **parameter** t.

For example, $x = a\cos\theta$, $y = a\sin\theta$ describe the circle $x^2+y^2 = a^2$. Here θ is a parameter. Similarly, $x = at^2$, $y = 2at$ describe the parabola $y^2 = 4ax$ in terms of the parameter t.

We shall now find the derivative of a function $y = f(x)$, where $x = f(t)$, $y = \phi(t)$.

We assume that $x = f(t)$ admits an inverse function $t = \psi(x)$. Then $y = \phi(t) = \phi[\psi(x)]$, so that y is a function of x. By the chain rule for composite functions we can then write

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} \text{ or } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Therefore, to find $\frac{dy}{dx}$, we shall find $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and take the quotient as shown in the following examples.

Example 16 : Find $\frac{dy}{dx}$ given $x = at^2$, $y = 2at$.

Solution : $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$.

Therefore, $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$.

Example 17 : If $x = a(\theta-\sin\theta)$; $y = a(1-\cos\theta)$. Find $\frac{dy}{dx}$ at $\theta = \pi/4$.

Solution : $\frac{dx}{d\theta} = a(1-\cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$

Therefore, $\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)}$,

and at $\theta = \pi/4$, $\frac{dy}{dx} = \frac{\sin\pi/4}{(1-\cos\pi/4)} = \frac{1/\sqrt{2}}{(1-1/\sqrt{2})}$

$=1/(\sqrt{2}-1)$.

You can now try this exercise.

E16) Find $\frac{dy}{dx}$ for the following functions :

a) $x = a\cos\theta$, $y = a\sin\theta$

b) $x = t^2/2$, $y = t^3/3$

c) $x = az$, $y = a/z$

d) $x = a\cos^3 t$, $y = a\sin^3 t$

In Unit 1 you have already read that any 1–1 function is invertible. That is, if f is a 1–1 function then it has an inverse function which we denote by f^{-1}. In this case the domain and range of the function f will be respectively the range and domain of the function f^{-1}. If f is a differentiable function of x, then the inverse function f^{-1} if it exists is in general a differentiable function. In the following section we shall show how to differentiate an inverse function.

6.7 DIFFERENTIATION OF INVERSE FUNCTIONS

Suppose we are given a function $y = f(x) = x$, then f has an inverse function given by $f^{-1}(x) = x$; but if $y = f(x) = x^2$ then since it cannot be solved uniquely for x, the function is not 1–1. Therefore, it does not have an inverse. Similarly, sin x does not have an inverse if $x \in \mathbf{R}$. It is evident from Fig. 6 that sin x has an inverse in the interval $[-\pi/2, \pi/2]$.

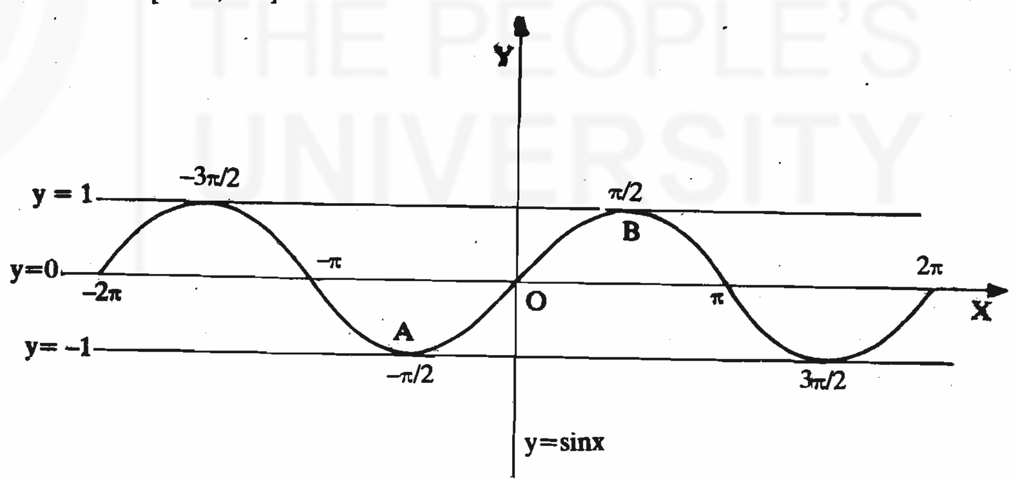


Fig. 6

Remember that if a function is 1–1 then no line parallel to the x-axis (including the x-axis) will cut the graph in more than one point.

Let us consider a function $y = f(x)$. We assume that it admits of an inverse function $x = \phi(y)$. We then have to find a relation between $f'(x)$ and $\phi'(y)$. Suppose that δy be the increment in y corresponding to increase δx in x. Then

$\frac{\delta x}{\delta y} \cdot \frac{\delta y}{\delta x} = 1$, since $\frac{\delta x}{\delta y}$ and $\frac{\delta y}{\delta x}$ are quotients.

Now, $\frac{\delta x}{\delta y} = \frac{1}{\delta y/\delta x}$ and let $\delta x \to 0$. Then as $\delta x \to 0$, $\delta y \to 0$ and

we have $\frac{dx}{dy} = \frac{1}{dy/dx}$

or $\frac{dx}{dy} \cdot \frac{dy}{dx} = 1$, that is $f'(x)\cdot\phi'(y) = 1$.

Thus, dy/dx and dx/dy are reciprocal to each other. For instance, consider a function

$y = f(x) = 2x+1$, then $\frac{dy}{dx} = 2$.

Also, $x = \frac{y-1}{2}$ is its inverse function and $\frac{dx}{dy} = \frac{1}{2}$; and

$\frac{dy}{dx} \cdot \frac{dx}{dy} = 2 \cdot \frac{1}{2} = 1$

You can now try the following exercises.

E17) If $y = x^{3/2}$ for $x > 0$, differentiate the inverse function and verify that

$$\frac{dx}{dy} = \left[\frac{dy}{dx}\right]^{-1}$$

E18) If $y = x^2+5x+8$, find $\frac{dx}{dy}$.

Continuing further with the differentiation of inverse functions we now obtain the derivatives of inverse trigonometric functions.

Differentiation of Inverse Trigonometric Functions

The trigonometric functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$ are generally defined as the inverse of the corresponding trigonometric functions. To fix ideas the discussions in this section are restricted to $\sin^{-1}x$ but, similar considerations apply to other inverse trigonometric functions. You know that $\sin^{-1}x$ is defined as the angle whose sine is x. The sign (−1) here does not denote the power of the function. It is only a symbol to denote the inverse of the function. However, $\sin^{-1}x$ because of the way it has been defined is infinitely many valued. To explain this we consider the functional equation

$$x = \sin y \quad \ldots\ldots\ldots \quad (1)$$

Now to each value of y, there corresponds just one value of x in (1). On the other hand, the same value of x corresponds to an unlimited number of values of the angle since we can find an unlimited number of values of the angle y, whose sine is x. For example, corresponding to $x = 0$ we have $y = 0, \pi, 2\pi, \ldots$. Thus, $\sin^{-1}x$, as defined above is not unique. The same remark applies to the remaining trigonometric functions also. To avoid this, these functions can be defined only on the intervals in which the corresponding trigonometric functions are uniquely defined. The values in these intervals are called their **principal values.** For example, sin y is 1−1 function in $[-\pi/2, \pi/2]$ whose range is $[-1,1]$. Consequently $\sin^{-1}x$ is uniquely determined for each x in $[-1,1]$. Here, we shall be finding the derivatives of all these inverse trigonometric functions corresponding to their principal values.

Derivative of $\sin^{-1}x$: $(-\pi/2 < \sin^{-1}x < \pi/2)$.

Let $y = \sin^{-1}x$. By definition $x = \sin y$.

Therefore $\frac{dx}{dy} = \cos y = +\sqrt{1-\sin^2 y} = +\sqrt{1-x^2}$, the positive sign is taken because y lies between $-\pi/2$ and $\pi/2$ where cos y is +ve.

Thus, $\frac{dy}{dx} = 1(dx/dy) = 1/\sqrt{1-x^2}$

or $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$.

Derivative of $\cos^{-1}x$: $(0 < \cos^{-1}x < \pi)$

If $y = \cos^{-1}x$, $x = \cos y$ and $\frac{dx}{dy} = -\sin y$

Therefore, $\frac{dx}{dy} = -\sqrt{1-x^2}$; again +ve root of sin y has been taken because sin y i positive between 0 and π.

Thus, $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$.

Derivative of $\tan^{-1}x$: ($-\pi/2 < \tan^{-1}x < \pi/2$)

If $y = \tan^{-1}x$, then $x = \tan y$ and $\frac{dx}{dy} = \sec^2 y$.

So, $\frac{dx}{dy} = 1+\tan^2 y = 1+x^2$. Thus, $\frac{dy}{dx} = 1/(1+x^2)$.

or $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$.

Derivative of $\cot^{-1}x$: ($0 < \cot^{-1}x < \pi$).

If $y = \cot^{-1}x$, then $x = \cot y$, $dx/dy = -\text{cosec}^2 y$.

Therefore, $\frac{dy}{dx} = \frac{-1}{\text{cosec}^2 y} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$

or $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$.

Similarly, we have $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$.

and $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$.

We now sum up the above six results in Table 3.

Table 3

f(x)	$\frac{df}{dx}$
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1}x$	$\frac{1}{1+x^2}$
$\cot^{-1}x$	$\frac{-1}{1+x^2}$
$\sec^{-1}x$	$\frac{1}{x\sqrt{x^2-1}}$
$\text{cosec}^{-1}x$	$\frac{-1}{x\sqrt{x^2-1}}$

Using these formulas you can now very easily do this exercise.

E19) Differentiate the following :

a) $\sin^{-1}2x$ b) $\cos^{-1}3x/4$ c) $\tan^{-1}(x^2+1)$

d) $x\sin^{-1}x$ e) $\tan^{-1}(x/a)$

So far we have obtained several results that enable us to compute derivatives of many important functions. In each case the function f whose derivative we sought was defined by an explicit formula, such as $f(x) = x^2+4$, $f(x) = x\sin x$, and so on.

We now consider a method, known as implicit differentiation, by which we can compute the derivative of a function without having an explicit formula for the function.

6.8 DIFFERENTIATION OF IMPLICIT FUNCTIONS AND LOGARITHMIC DIFFERENTIATION

The expressions $x^2+2ax+by^2 = 1$ or $\sin(x+y) = 1$ are examples of implicit functions. These are of the form $f(x,y) = 0$.

In each of these y is not expressed directly in terms of x but only a functional relation involving x and y is given.

To obtain the derivative of an implicit function, we first try to express $f(x,y) = 0$ as $y = \phi(x)$, if it is possible and then the derivative can be obtained as in the case of explicit functions.

But, if it is not possible to transform the given implicit function $f(x,y) = 0$ to a form $y = \phi(x)$, then we differentiate the given expression term by term with respect to x and get the value of dy/dx from this. To get a clear idea of what has been said above let us consider the following examples.

Example 18 : Given $x^2+y^2=a^2$, find $\frac{dy}{dx}$.

Solution : We can rewrite y as $y = \pm\sqrt{a^2-x^2}$ and then

$$\frac{dy}{dx} = \pm\frac{1}{2}(a^2-x^2)^{-1/2}(-2x) = \mp x(a^2-x^2)^{-1/2}.$$

Example 19 : Given the relation $x^3+y^3-6xy=0$, find dy/dx.

Solution : Here we cannot proceed as in the above example. Therefore, differentiating the given expression term by term with respect to x, we get,

$$3x^2 + 3y^2\frac{dy}{dx} - 6y\cdot 1 - 6x\frac{dy}{dx} = 0$$

Therefore, $\frac{dy}{dx}(3y^2-6x) = 6y - 3x^2 = 3(2y-x^2)$

or $\frac{dy}{dx} = \frac{3(2y-x^2)}{3(y^2-2x)}$.

Note that y^3 is a function of x, therefore, $\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\cdot\frac{dy}{dx} =$

$3y^2\cdot\frac{dy}{dx}$ and similarly for other y terms in the problem.

While differentiating the given expression term by term, remember that every time you differentiate a function of y, it has to be multiplied by dy/dx in order to get the derivative with respect to x.

Example 20 : $e^y = xy$, find dy/dx.

Solution : $e^y\frac{dy}{dx} = y + x\cdot\frac{dy}{dx}$,

therefore, $\frac{dy}{dx}(e^y-x) = y$ giving $\frac{dy}{dx} = \frac{y}{e^y-x}$.

Example 21 : $\ln(xy) = x^2+y^2$, find $\frac{dy}{dx}$.

Solution : $\frac{1}{xy}\left[x\frac{dy}{dx} + y\cdot 1\right] = 2x + 2y\frac{dy}{dx}$

or $\frac{dy}{dx}(1/y - 2y) = 2x - 1/x$, thus, $\frac{dy}{dx} = \frac{2x - 1/x}{1/y - 2y}$

You can now try this exercise.

E20) Find dy/dx in the following cases.

a) $x^2+y^2=a^2$ b) $y^2 = 4ax$ c) $xy = k^2$

d) $x^{1/2} + y^{1/2} = a^{1/2}$ e) $e^x = xy$

When the function to be differentiated is the product of a number of factors then we make use of the technique called logarithmic differentiation.

Method of differentiating functions by first taking logarithms and then differentiating is called logarithmic differentiation.

Logarithmic differentiation

When using the technique of logarithmic differentiation we make use of certain properties of logarithms which facilitate algebraic manipulations. For example, the following properties of logarithmic functions are known : (see Unit 2)

1) If $u = xyz/abc$, then $\ln u = \ln x + \ln y + \ln z - \ln a - \ln b - \ln c$ a complicated operation with products and quotients transformed into a simple operation involving sums and differences.

2) If $y = a^f$ then $\ln y = f \ln a$, where a is a constant. A power function has been expressed in a simple form.

3) If $y = u^v$, where u and v are both functions,

$\ln y = v \ln u$ (a simpler version of the equation).
Some of these properties can be used with advantage when differentiating certain types of functions and this will be illustrated, through these examples.

Example 22 : $y = a^{f(x)}$, a being a constant, find $\frac{dy}{dx}$.

Solution : Take logarithm of both sides, then
$\ln y = f(x)\cdot\ln a$ (ln a is again constant because a is constant). Differentiating with respect to x, we have

$$\frac{1}{y}\frac{dy}{dx} = \ln a\cdot f'(x)$$

Therefore, $\frac{dy}{dx} = y\cdot\ln a\cdot f'(x) = a^{f(a)}\ln a\cdot f'(x)$.

Example 23 : If $y = u^v$, where u and v are functions of x, find dy/dx.

Solution : Taking logarithms of both sides $\ln y = v \ln u$. Now differentiating both sides with respect to x,

$$1/y\frac{dy}{dx} = v\frac{d}{dx}(\ln u) + \ln u\cdot\frac{d}{dx}(v)$$

$$= v\cdot 1/u\frac{du}{dx} + \ln u\cdot\frac{dv}{dx}$$

Therefore, $\frac{dy}{dx} = y[\frac{vdu}{udx} + \ln u\frac{dv}{dx}]$

$$= u^v[\frac{vdu}{udx} + \ln u\frac{dv}{dx}]$$

E21) Find $\frac{dy}{dx}$ where,

a) $y = a^{e^x}$, b) $y = 2^{-3x}$, c) $y = e^{x^2}$, d) $y = (2x)^{\sin x}$

6.9 PHYSICAL ASPECT OF DERIVATIVE

In this section we will discuss a very commonly used aspect of the derivative, namely, the derivative as a rate-measurer. Indeed, this interpretation of the derivative is inherent in the definition itself. Recall that, $\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x}$.

Now $\delta y/\delta x$ is the **rate of change** of y with respect to x in the interval δx. Therefore, dy/dx can be seen as the limiting value of the rate when $\delta x \to 0$. We call this the **instantaneous rate of change** of y with respect to x. For example, if a car moves a distance δs in time δt then the rate of change of distance with respect to time, that is, the speed of the car is $\delta s/\delta t$ and we say that the **velocity** v of the car is ds/dt at any instant. If the car is not moving with constant velocity, then d/dt (ds/dt) =

$d^2s/dt^2 = \frac{dv}{dt}$ is called its **acceleration** a at any instant.

The derivative d^2s/dt^2 is called the second order derivative of s with respect to t. We shall be studying about the second and higher order derivatives of a function in Unit 7.

We now consider two examples which establish the 1–1 correspondence between the mathematical problem expressed in terms of derivatives and its physical interpretation.

Example 24 : If the law of motion of a particle is given as

$s = 25+3t^2-t^3$, then

a) find its velocity and acceleration.

b) find the distance covered by the particle in time t = 2 units.

Solution : a) Since the law of motion is given by $s = 25+3t^2-t^3$,

its velocity v $= \frac{ds}{dt} = 6t - 3t^2$ and

acceleration a $= \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = 6 - 6t.$

b) The initial distance of the particle that is, the distance at $t = 0$ is $25 + 3(0)^2 - (0)^3 = 25$ units. The distance at t = 2 is $25 + 3.(2^2) - (2^3) = 25 + 12 - 8 = 29$ units.
Hence, the distance covered by the particle in t = 2 units is 29 – 25 = 4 units.

Note that in Example 24 the velocity $v = 6t-3t^2$ of the particle is zero at t = 0 and at t = 2. Physically this fact can be interpreted by saying that the particle started from rest and comes to instantaneous rest after time t = 2 units.

Example 25 : Water is dropping from a burette on a clean circular container plate, so that a circular pool is formed. The pool is gradually increasing in area and in radius. What is the ratio of the increase in area to the increase in radius. What is the numerical value of this ratio when the radius is 2 cms.

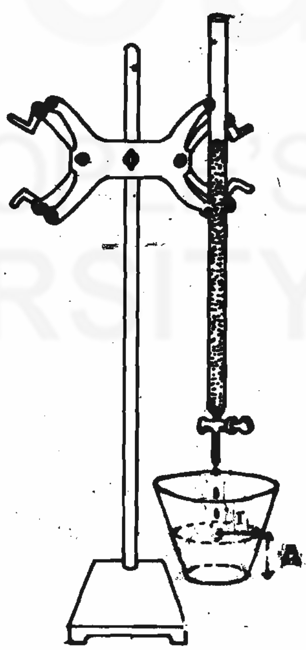


Fig. 7

Solution : If A is the area of the circular pool (see Fig. 7), then, $A = \pi r^2$, where r is the radius of the circle.
Therefore, $dA/dr = 2\pi r$ gives the instantaneous ratio of the increase in area to the increase in radius. Numerical value of this rate of change when r = 2 cms is $(dA/dr)_{r=2} = 2\pi.2 = 4\pi$.

You can now try the following exercises.

E22) If a protein of mass m disintegrates into amino acids according to the formula $m = 28/(t+2)$ where t indicates time, find the average rate of reaction in the time interval t = 0 to t = 2.

E23) If a metabolic experiment shows that the mass m of glucose decreases with respect to time according to the equation $m = (4.5) - (.03)t^2$, find the rate of reaction at t = 2.

E24) Vander Waal's equation for a gas is $(p+av^{-2})(v-b) = k$ where p is the pressure and v is the volume of gas and a, b and k are constant. What is the rate of change in volume with respect to change in the pressure?

6.10 SUMMARY

We conclude this unit by giving a summary of what we have covered in this unit.

1) The definition of the limit of a function.
2) The definition of the continuity of a function at a point.
3) If $y = f(x)$ then the derivative $dy/dx = \lim_{\delta x \to 0} \delta y/\delta x$
   $= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$.
4) $d/dx(x^n) = nx^{n-1}$, where n is any real number.
5) Geometrically, dy/dx represents the slope of the tangent to the curve $y = f(x)$ at the point (x,y).
6) d/dx(constant) = 0.
7) $d/dx[cf(x)] = c\, df/dx$ where c is a constant.
8) $d/dx[f_1(x) \pm f_2(x)] = df_1/dx \pm df_2/dx$.
9) $d/dx(uv) = u\, dv/dx + v\, du/dx$, where u and v are functions of x.
10) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$ where u and v are functions of x.
11) $d/dx[f(\phi(x))] = df/d\phi \cdot d\phi/dx$.
12) $d/dx(e^x) = e^x$.
13) $d/dx(\ln x) = 1/x$.
14) $d/dx(q^x) = q^x \ln q$, where q is a positive constant.
15) $\frac{dy}{dx} = \frac{1}{dx/dy}$.
16) If $x = f(t)$, $y = \phi(t)$ then
    $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.
17) Differentiation of functions expressed in the form $f(x,y) = 0$.

6.11 SOLUTIONS/ANSWERS

E1) (a) 1/7 (b) 3/7 (c) does not exist (d) 1 (e) 2x.

E2) (a) $x = \pm 3$ (b) no value (c) $x = -2$.

E3) $f(x) = 16$.

E4) Function is not continuous at $x = -2$. Also, it cannot be redefined at $x = -2$, because the function does not have a finite limit at $x = -2$.

E5) (a) $3x^2$ (b) $-1/x^2$ (c) $\frac{3}{2} x^{1/2}$

E6) (a) 1 (b) $-8/r^3$ (c) $1/3\, x^{-2/3}$ (d) $2/5\, t^{-3/5}$ (e) $12u^{11}$
(f) $-9x^{-10}$

E7) Slope of graph $= \frac{dy}{dx} = x^2-2x-2$

Slope of $x-y-6 = 0$ is 1.
We wish x, for which $x^2-2x-2 = 1$
This gives, $x = 3$ or -1.
Corresponding to $x = 3$, $y = -9$ and for $x = -1$, $y = -7/3$.
Thus, there are two points $(3, -9)$ and $(-1, -7/3)$.

E8) (a) $15x^4 + x^{-1/2}$ (b) $1 - 4x^{-2} + 44x^{-5}$ (c) $3/2\, x^{1/2} + 1/2\, x^{-1/2} - 1/2\, x^{-3/2}$

E9) (a) $3/5\, v^{-2/5} - 6$ (b) $18s^5 - 20s^3 + 12s$.

E10) (a) $(x^2+7)(-1) + (3-x)(2x)$ (b) $(x^3+3x)\, 2x + (x^2+2)(3x^2)$.
(c) $(x^3 - x + 2)\, 2x + (x^2 - 1)(3x^2-1)$.

E11) (a) $\frac{23}{(3x+4)^2}$ (b) $\frac{18}{(2x^2+3)^2}$ (c) $\frac{2}{(x+1)^2}$ (d) $1 + \frac{2-x^2}{(x^2+2)^2}$

E12) (a) $x^3\cos x + 3x^2\sin x$ (b) $\cos^2 x - \sin^2 x$

(c) $\frac{3\cos x + (3x+2)\sin x}{\cos^2 x}$ (d) cosec x − xcot x cosec x (e) 4sin x cos x

E13) (a) 3cos 3x (b) −5sin 5x (c) $2x\tan x + \frac{x^2}{1+x^2}$

(d) $\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x$ (e) $\frac{6(x+1)(2x-3)^2 - (2x-3)^3}{(x+1)^2}$

(f) $\frac{-1}{2v\sqrt{x}}\sin\sqrt{x}$. (g) $\frac{(1+\cos x)\sin 2x - \sin^3 x}{(1+\cos x)^2}$.

E14) (a) $3^{-5x}\ln 3\,(-5)$ (b) $e^{\sin\theta}\cos\theta$ (c) $(-2)e^{-2r}$ (d) $2se^{s^2}$

(e) $e^{x\cos x}[\cos x - x\sin x]$.

E15) (a) $\frac{2u}{u^2-3}$ (b) $\frac{t+1}{-t}$ (c) $\frac{1}{2x}$ (d) $\frac{\cot\sqrt{\theta}}{2\sqrt{\theta}}$ (e) $\frac{2x\ln x - x}{(\ln x)^2}$

(f) $\frac{1}{r} + \frac{2}{r\ln(r^2)}$

E16) (a) −cot θ (b) t (c) $\frac{1}{z^2}$ (d) − cot t.

E17) $\frac{dy}{dx} = 3/2\, x^{1/2}$, $\frac{dx}{dy} = 2/3\, y^{-1/3}$. Hence the result.

E18) $\frac{1}{2x+5}$

E19) (a) $\frac{2}{\sqrt{1-x^2}}$ (b) $\frac{-3}{\sqrt{16-9x^2}}$ (c) $\frac{2x}{1+(x^2+1)^2}$

(d) $\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}$ (e) $\frac{a}{a^2+x^2}$

E20) (a) −x/y (b) 2a/y (c) −y/x (d) $\sqrt{y/x}$ (e) $\frac{e^x-y}{x}$

E21) (a) $a^{e^x}e^x\ln a$ (b) $(-2)^{3x}\cdot 3\ln 2$ (c) $2xe^{x^2}$

(d) $(2x)^{\sin x}\left[\frac{\sin x}{x} + \cos x\ln(2x)\right]$

E22) Rate of reaction is $\frac{dm}{dt} = \frac{-28}{(t+2)^2}$

$\left.\frac{dm}{dt}\right|_{t=0} = -7$, $\left.\frac{dm}{dt}\right|_{t=2} = -1.75$

average rate of reaction in the time interval t = 0

to t = 2 is $\frac{1}{2}\left[\left.\frac{dm}{dt}\right|_{t=0} + \left.\frac{dm}{dt}\right|_{t=2}\right] = \frac{-7-1.75}{2} = -4.375$.

E23) $\left.\frac{dm}{dt}\right|_{t=2} = .12$

E24) Differentiate the implicit function with respect to p and get

$$\frac{dv}{dp} = \frac{v^3(b-v)}{pv^3 - av + 2ab}$$

UNIT 7 APPLICATIONS OF DIFFERENTIAL CALCULUS

Structure

7.1 INTRODUCTION

In Unit 6, we obtained derivatives of various types of functions and also derived rules for obtaining these derivatives. In this unit the main emphasis will be on the applications of derivatives.

In all branches of science we often face problems like : (i) How can we find accurate values of a function f corresponding to given values of x? (ii) How can we find the maximum and minimum values of a function f in a certain domain? A simple way of tackling such problems is through the application of differentiation. Consider, for instance, the first question above. One of the ways of calculating functional values to a certain degree of accuracy is through the expansion of the function as in a power series. The method of Maclaurin expansion is one such technique, which has been explained in this unit. We shall also explain the process of finding second derivatives, the maxima and minima of a function and of tracing a given curve. We shall also discuss functions of two variables.

To start with, we have talked about the problem of finding tangents and normals to a given curve, which are geometrical applications of differentiation.

Objectives

After reading this unit, you should be able to

- write the equation of the tangent and the normal to a given curve at a given point,
- compute the second and higher order derivatives of a given function,
- write the power series expansion of some functions,
- compute the maxima and minima of various functions,
- identify and draw the graphs of some significant curves,
- find the first and second order partial derivatives of a function of two variables given in explicit, implicit or parametric form.

7.2 TANGENTS AND NORMALS

You have already studied in Unit 6, that if a curve is given by the equation $y = f(x)$ where $f(x)$ has a derivative $f'(x)$ at every point in the domain of f, then