UNIT 5 VECTORS

Structure

5.1 INTRODUCTION

The study of physical problems has most often led to new mathematical concepts. These concepts provide a simple way of describing the relevant physical situations. One such concept is that of vectors. You might have noticed that in the study of physical problems you normally come across two types of quantities:

(a) those that can be completely specified by their magnitude. We call such quantities **scalars.** For instance, the number 212 on the Fahrenheit thermometer specifies the boiling temperature of water at standard pressure. Temperature is then a scalar quantity. Other such quantities are mass, volume, density etc.

(b) those that cannot be described by their magnitude alone. For a complete description of such quantities we need to know their **magnitudes** as well as their **directions.** Such quantities are called **vectors**. For example, the instantaneous velocity of a train is not fully described by stating that its speed is 100 mph. If one also designates the direction of the motion (say by stating that it heads northeast), then the velocity is unambiguously described. Velocity is thus an example of a vector quantity. The other simple example of a vector is the displacement of a particle from a position P to a new position P'.

Mathematical tools for dealing with directed quantities in three dimensions are outstanding products of nineteenth century mathematics. The algebra of vectors was initiated principally through the works of W.R. Hamilton and H.G. Grassman in the middle of the 19th century and brought to the form presented here by the efforts of O. Heaviside and J.W. Gibbs in the late 19th century. It is true that any problem that can be solved by the use of vectors can also be treated by non-vectorial methods, but vector analysis is a shorthand which simplifies many calculations considerably. For instance, the problem of finding the area of a triangle, parallelogram etc. can be done very easily using vectors. The main advantage of using vectors is that magnitude and directions can be handled simultaneously. In this unit we shall be first introducing vectors as directed line segments and then develop the algebra of vectors. This will enable us to use vectors in the study of various physical and geometrical problems.

Objectives

After reading this unit you should be able to

- differentiate between vectors and scalars;
- represent vectors as directed line segments;
- obtain the sum, difference, scalar product and vector product of two or more vectors;
- use vectors in the study of simple geometrical and physical problems.

5.2 VECTORS AS DIRECTED LINE SEGMENTS

Any finite portion of a given straight line where the two end points are distinguished as **initial** point and **terminal** point, is called a directed line segment. The directed line segment whose initial point is A and terminal point is B, is denoted by the symbol $\overrightarrow{AB}$, that is, with an arrow overAB, pointing from A to B (Fig. 1).

Note that $\overrightarrow{AB}$ is not the same as $\overrightarrow{BA}$. This is because the initial point of $\overrightarrow{AB}$ is A, whereas that of $\overrightarrow{BA}$ is B. Thus, the two end points of a directed line segment are not interchangeable.

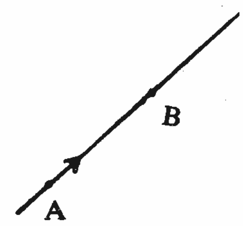


Fig.1 : Directed line segment

Associated with every directed line segment $\overrightarrow{AB}$, are its length, support and sense.

i) **Length :** The length of $\overrightarrow{AB}$ will be denoted by the symbol $|\overrightarrow{AB}|$. Clearly we have $|\overrightarrow{AB}| = |\overrightarrow{BA}|$.

ii) **Support :** The line of unlimited length of which a directed line segment is a part is called its line of support or simply the support.

iii) **Sense :** The sense of $\overrightarrow{AB}$ is from A to B and that of $\overrightarrow{BA}$ is from B to A so that the sense of a directed line segment is from its initial to the terminal point. The directed line segments $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same lengths and supports but different senses.

A **vector** is a directed line segment. Consider, for example, a force of 50 dynes acting on a particle in the direction $\overrightarrow{AB}$. In order to represent it by a line segment we select a scale, say, 10 dynes = 1 cm., and draw a line PQ of length 5 cm parallel to AB, and put an arrow over it pointing from P to Q. Then the directed line segment $\overrightarrow{PQ}$ represents the vector which in this case is the force of 50 dynes acting in the direction AB (Fig. 2).

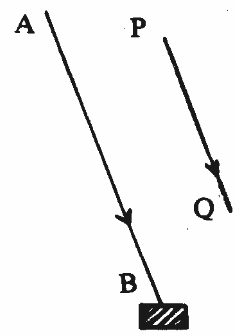
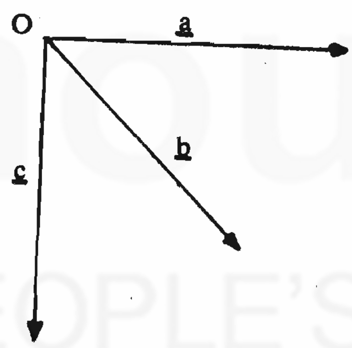


Fig. 2

In a similar way any vector quantity can be represented by a direct line segment. The length of the line segment gives the **magnitude** of the vector and the direction of the arrow gives the **direction** of the vector. Thus physically, a vector is a quantity having a magnitude and a direction. We shall be denoting vectors by bold face letters **a, b, ...** or by symbols $\overrightarrow{AB}$, $\overrightarrow{PQ}$ .... etc. **Magnitude of a vector a is usually denoted by |a|** This is read as modulus of **a** or, in short, mod **a**. If $\overrightarrow{AB}$ represents a vector **a**, then | **a** | = the length AB. Further, if A denotes a point (x_1,y_1,z_1) and B denotes (x_2,y_2,z_2), then the components of vector a represented by $\overrightarrow{AB}$ are $(x_2 - x_1)$, $(y_2 - y_1)$ and $(z_2 - z_1)$.

A vector of unit length is called a **unit vector.** If **a** is any vector with $|\mathbf{a}| > 0$, then $\dfrac{\mathbf{a}}{|\mathbf{a}|}$ is a unit vector having the same direction as **a.** A vector whose initial and terminal points are coincident is called the **zero vector.** The length of the zero vector is zero but it can be thought of as having any line as its line of support.



(a)

Equality of Vectors : Vectors **a** and **b** are said to be equal if they have the same magnitude and direction and this may be expressed by writing

a = b

If $\overrightarrow{PQ}$ represents a vector **a,** then any line segment P′Q′ parallel to PQ and having the same length and direction will also represent the same vector. Conversely, if

$\overrightarrow{PQ} = \overrightarrow{P'Q'}$

then it implies that

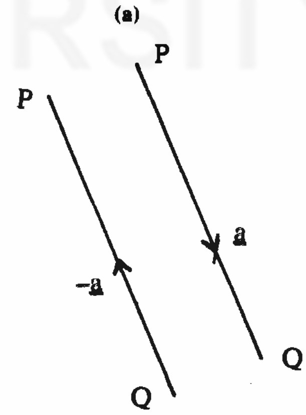
i) the length PQ = the length P′Q′, and

ii) the line PQ $||$ the line P′Q′.

Vectors with the same initial point are called **co-initial vectors.**

Thus, vectors **a, b, c,** in Fig. 3(a) are co-initial.

A vector having the same magnitude as **a** but whose direction is opposite to that of **a,** is defined as the **negative** of **a,** and is denoted by **–a.** If $\overrightarrow{PQ}$ = **a** then $\overrightarrow{QP}$ = **–a** (Fig. 3(b)).



(b)

Fig. 3(a) Co-initial vectors
(b) Negative of a vector

If p is a number and **a** is a vector, then p**a** is defined as a vector parallel to **a** whose magnitude is $|p|$ times the magnitude of **a.** Here $|p|$ stands for the absolute value of p. If $p > 0$, then p**a** has the same direction as **a.** If $p < 0$, p**a** is in a direction opposite to **a.** For example, 2**a** is a vector parallel to **a** and having magnitude equal to twice the magnitude of **a.** Similarly, $-\frac{1}{2}\mathbf{a}$ is a vector parallel to **a** but in the opposite sense and its magnitude is half of the magnitude of **a.** (see Fig. 4)

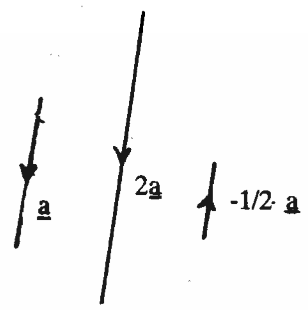


Fig. 4 Multiplication of vectors by numbers

Collinear Vectors : Vectors having the same or parallel supports are called collinear vectors. Collinear co-initial vectors have the same support. The vectors **a** and p**a** are collinear.

Now if you have understood the above discussion then you will find the following exercise very easy.

E1) Find the component of **a** with given initial point $P(x_1,y_1,z_1)$ and terminal point $Q(x_2,y_2,z_2)$.

a) P(0, 0 ,1), Q(1 ,1 ,1)

b) P(0, 0, −1), Q(0, 0 ,3)

c) P(2, 2 ,3), Q(2, 2, 3)

d) P(2, 3, 5), Q(4, 5 ,6)

Let us now develop an algebra of vectors which is very useful in the study of geometry and other branches of mathematics.

5.3 ALGEBRA OF VECTORS

By 'algebra of vectors', we mean various ways of combining vectors and scalars, satisfying different laws, called **laws of calculations.** Let us take these one by one.

5.3.1 Addition of Vectors

Addition of two vectors **a** and **b** is defined as follows:

Select any point P as the initial point and construct $\overrightarrow{PQ}$ = **a** (see Fig. 5(a)). Now take Q as the initial point and construct $\overrightarrow{QR}$ = **b**. Then the vector $\overrightarrow{PR}$ = **c** is defined as the sum of vectors **a** and **b** and we write it as

$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

or **c** = **a** + **b**

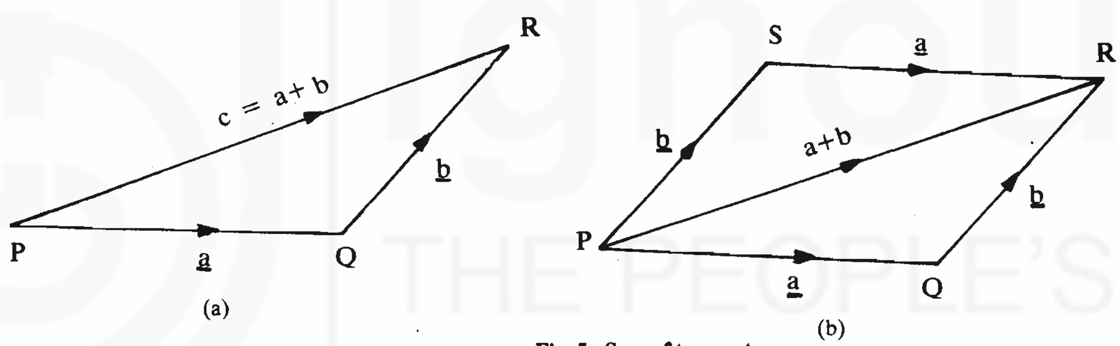


Fig. 5 : Sum of two vectors

Stated in words, if **a** and **b** are represented by two sides of a triangle taken in order, then the third side taken in the reverse order represents the sum of the vectors **a** and **b.** This is called the **trianlge law** of addition of vectors. (Fig. 5(a))

If the two vectors to be added are considered to be sides of a parallelogram, their sum can also be interpreted as given by the diagonal as shown in Fig. 5(b). This is called the **parallelogram law** of addition of vectors. Addition of more than two vectors can be carried out by repeated application of the rule for addition of two vectors. In Fig. 6. the sum of four vectors **a, b, c** and **d** is illustrated. The sum is the vector from the initial point A to the final point E. As can be seen from Fig. 6, the sum of **a** and **b** is $\overrightarrow{AC}$. The sum of $\overrightarrow{AC}$ and **c** is $\overrightarrow{AD}$. The sum of AD and **d** is $\overrightarrow{AE}$. Similarly, we can add any number of vectors.

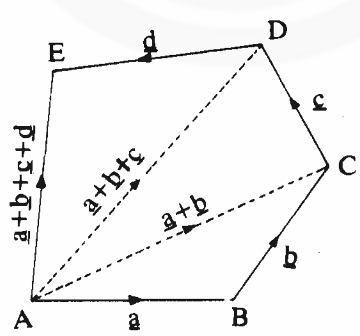


Fig. 6: Sum of more than two vectors

If two co-initial vectors **a** = $\overrightarrow{OA}$ and **b** = $\overrightarrow{OB}$ are to be added, then we draw a vector $\overrightarrow{AC}$ = **b,** with A as the initial point. In this case vector $\overrightarrow{OC}$ will represent the sum **a** + **b**.

It is clear from Fig. 5(b) that

$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SR}$

or **a** + **b** = **b** + **a**

Thus, **the addition of vectors is commutative.**
The **addition of vectors is also associative**; that is,

(a + b) + c = a + (b + c)

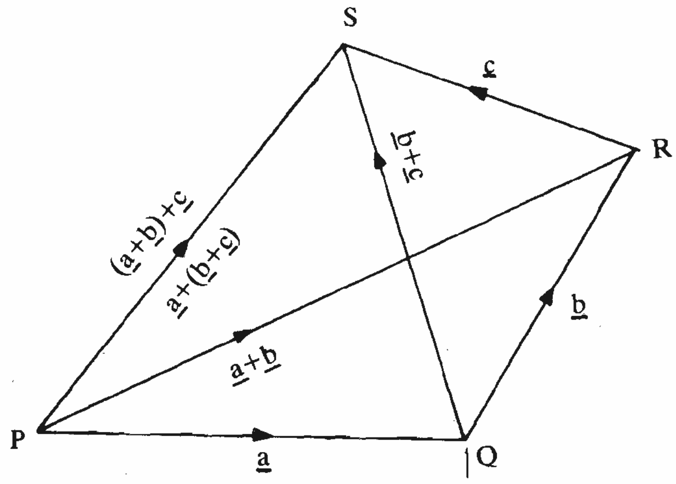


Fig. 7 : Associativity of Vector Addition

In Δ PQR (Fig. 7),

$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$.

or **a** + **b** = $\overrightarrow{PR}$ .... (1)

In Δ PRS, $\overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PS}$

or (**a** +**b**) + **c** = $\overrightarrow{PS}$

Again, in Δ QRS, $\overrightarrow{QR} + \overrightarrow{RS} = \overrightarrow{QS}$,

or **b** + **c** = $\overrightarrow{QS}$

In Δ PQS, $\overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}$

or **a** + (**b** + **c**) = $\overrightarrow{PS}$ .... (2)

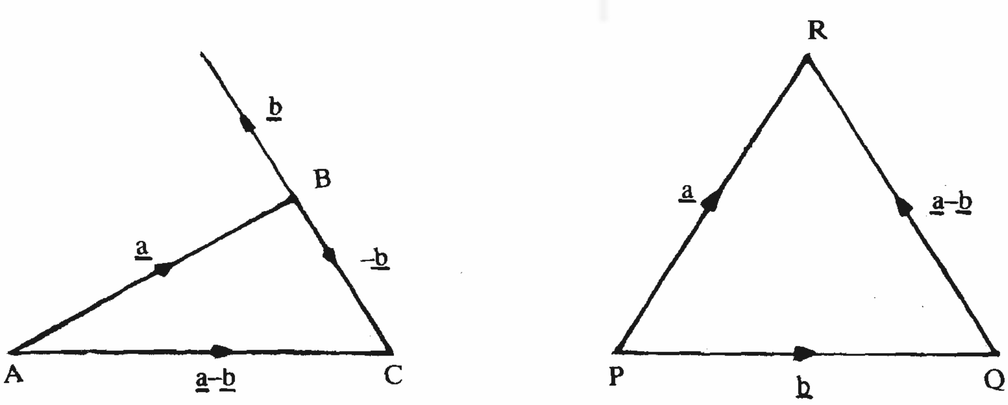
Thus, from (1) and (2), we get
(a + b) + c = a + (b + c).

Remark 1 : The sum of two vectors is zero if and only if they are of the same magnitude, their supports are parallel but their sense is opposite. For if, **a** + **b** = 0, then **a** = –**b** = p**b** (ref. Sec. 5.2) where p = –1
Thus, **a** and **b** have the same magnitude, are parallel but sense is opposite. Stated differently, the **sum of two non-parallel vectors can never be zero.**

Remark 2 : If x and y are any two scalars (numbers), and **a** and **b** two non-zero, non-parallel vectors, then x**a** + y**b** = 0 implies x = 0, y = 0. This is because x**a** is a vector parallel to **a** and y**b** is a vector parallel to **b**. Therefore, their sum cannot be zero, unless x = 0, y = 0.

5.3.2 Difference of Vectors

Subtraction of vector **b** vector **a** is defined by the equation **a** – **b** = **a** +(–**b**)



(a) **Fig. 8** (b)

In Fig. 8(a), $\vec{AC} = \vec{AB} + \vec{BC} = \mathbf{a} - \mathbf{b}$

In Fig. 8(b), $\vec{PQ} + \vec{QR} = \vec{PR}$

or $\mathbf{b} + \vec{QR} = \mathbf{a}$

or $\vec{QR} = \mathbf{a} - \mathbf{b}$.

Thus, we have, in particular

$\mathbf{a} - \mathbf{a} = \mathbf{a} + (-\mathbf{a}) = \mathbf{a} + (-1)\mathbf{a} = 0$

Let us now illustrate vector addition and subtraction with the help of the following examples.

Example 1 : ABCD is a quadrilateral in which $\vec{BC}$ is parallel to $\vec{AD}$ and the ratio of the lengths $\vec{BC}$: AD is 3 : 5. If $\vec{AB} = \mathbf{v}$ and $\vec{AD} = 5\mathbf{u}$, find the vectors represented by

i) $\vec{BC}$;   ii) $\vec{AC}$;   iii) $\vec{BD}$;   iv) $\vec{BC}$;   v) $\vec{AE}$

where E is on $\vec{BD}$ such that $\vec{BE} = 3/8\ \vec{BD}$.

Solution : (i) It is given that $\vec{BC}$ is parallel to $\vec{AD}$ and $\vec{BC}/\vec{AD} = 3/5$ or $\vec{BC} = 3/5\ \vec{AD}$. Hence $\vec{BC} = \frac{3}{5}\vec{AD} = 3\mathbf{u}$. (Fig. 9).

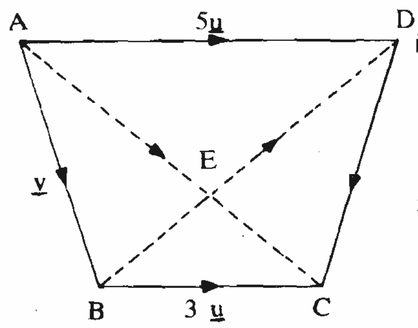


Fig. 9

ii) In ΔABC,

$\vec{AB} + \vec{BC} = \vec{AC}$

or, $\vec{AC} = \mathbf{v} + 3\mathbf{u}$.

iii) In ΔABD,

$\vec{AB} + \vec{BD} = \vec{AD}$

or, $\mathbf{v} + \vec{BD} = 5\mathbf{u}$.

or, $\vec{BD} = 5\mathbf{u} - \mathbf{v}$.

iv) In Δ ADC,

$\vec{AD} + \vec{DC} = \vec{AC}$

or, $5\mathbf{u} + \vec{DC} = \mathbf{v} + 3\mathbf{u}$

or $\vec{DC} = \mathbf{v} + 3\mathbf{u} - 5\mathbf{u}$

$= \mathbf{v} - 2\mathbf{u}$.

v) $\vec{BE} = \frac{3}{8}\vec{BD}$

Therefore, $\vec{BE} = \frac{3}{8}\vec{BD}\ (5\mathbf{u} - \mathbf{v})$

In ΔABE (Fig. 9), $\vec{AB} + \vec{BE} = \vec{AE}$

or $\vec{AE} = \mathbf{v} + \frac{3}{8}(5\mathbf{u} - \mathbf{v})$

$= \frac{15}{8}\mathbf{u} + \frac{5}{8}\mathbf{v}$

You must have proved a number of theorems in geometry in your high school. Many of these can be very easily proved by using vector addition and subtraction. In the next two examples you will see an illustration of this.

Example 2 : Prove, with the help of vectors, that diagonals of a parallelogram bisect each other.

Solution : Consider the parallelogram OABC (Fig. 10).

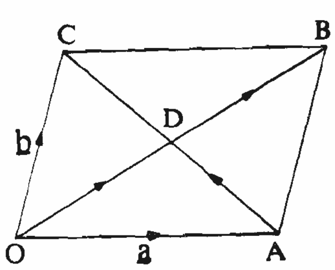


Fig. 10

Let $\vec{OA} = \mathbf{a}$, $\vec{OC} = \mathbf{b}$.

Then, $\vec{OB} = \vec{OA} + \vec{AB} = \mathbf{a} + \mathbf{b}$

$\vec{OC} = \vec{OA} + \vec{AC}$ or $\vec{AC} = \mathbf{b} - \mathbf{a}$.

Let D be the point of intersection of the diagonals. Then vector $\vec{OD}$ is some fraction of $\vec{OB}$. Let $\vec{OD} = m\ \vec{OB} = m\ (\mathbf{a} + \mathbf{b})$ for some m. Also, the vector $\vec{AD}$ is a part of $\vec{AC}$. Let $\vec{AD} = n(\mathbf{b} - \mathbf{a})$.

Now in ΔOAD,

$\vec{OA} + \vec{AB} = \vec{OD}$

or $\mathbf{a} + n\ (\mathbf{b} - \mathbf{a}) = m\ (\mathbf{a} + \mathbf{b})$

or $(1 - n - m)\mathbf{a} + (n - m)\,\mathbf{b} = 0$

since **a** and **b** are non-parallel vectors we must have

$1 - n - m = 0$ and $n - m = 0$ (see Remark 2)

which gives, $n = m = \frac{1}{2}$.

Thus, $\overrightarrow{OD} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2}\overrightarrow{OB}$

$$\overrightarrow{AD} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\overrightarrow{AC}.$$

Hence, the diagonals are bisected at D.

Example 3 : Prove that the straight line join the mid points of two sides of a triangle is parallel to the third side and its length is half of that of the third side.

Solution: Let $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ as shown in Fig. 11. Since D and E are the mid points of BC and AC respectively, we have

$\overrightarrow{DC} = \frac{1}{2}\mathbf{b}$ and $\overrightarrow{EC} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

Let $\overrightarrow{ED} = \mathbf{c}$. Then, in ΔDEC,

$\overrightarrow{ED} + \overrightarrow{DC} = \overrightarrow{EC}$

or, $\mathbf{c} + \frac{1}{2}b = \frac{1}{2}(a + b)$

or, $\mathbf{c} = \frac{1}{2}a$ or $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AB}$

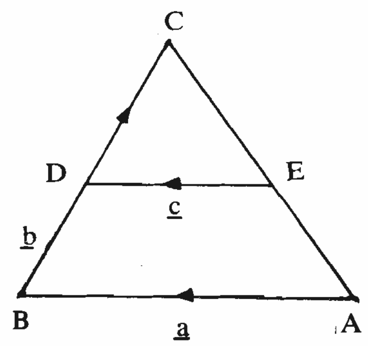


Fig. 11

Thus, Ed is parallel to AB and its length is equal to half that of AB.

You may now try the following exercises.

E2 ABCD is a quadrilateral in which BC is parallel to AD and the ratio of lengths BC : AD is 4 : 7. If $\overrightarrow{AB} = \mathbf{v}$ and $\overrightarrow{AD} = 7\mathbf{u}$.
Find the following vectors in terms of **u** and **v**.

(a) $\overrightarrow{BC}$ (b) $\overrightarrow{AC}$ (c) $\overrightarrow{BD}$ (d) $\overrightarrow{DC}$ (e) $\overrightarrow{AE}$

where E is a point on BD such that BE = $\frac{4}{11}$ BD in length.

E3) ABC is any triangle and D, E, F are the mid points of its sides BC, CA, AB respectively. Express

a) $\overrightarrow{BC}$, $\overrightarrow{AD}$, $\overrightarrow{BE}$, $\overrightarrow{CF}$, in term of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AD}$, in term of $\overrightarrow{BE}$ and $\overrightarrow{CF}$.

c) $\overrightarrow{AC}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$, $\overrightarrow{CF}$, in term of $\overrightarrow{AB}$ and $\overrightarrow{BE}$.

d) Prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = 0$.

E4) ABCD is a parallelogram and AC, BD are its diagonals.
Express

a) $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

b) $\overrightarrow{AB}$ and $\overrightarrow{AD}$ in terms of $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

c) $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of $\overrightarrow{AD}$ and $\overrightarrow{BD}$.

E5) Show by vector methods that the line joining a vertex of a parallelogram with the mid point of an opposite side cuts a diagonal at a point of trisection.

5.3.3 Resolution of a Vector

We consider three mutually perpendicular axes OX, OY and OZ as in Fig. 12. The coordinate axes OX, OY, OZ form a **right-handed system** provided a screw pointing along OZ advances when it is twisted from OX to OY. Define the vector **i** to be a vector of unit length pointing in the direction of the positive x-axis, the vector **j** to be a vector of unit length pointing in the direction of the positive y-axis and the vector **k** to be a vector of unit length pointing in the direction of the positive z-axis. Then the unit vectors **i, j, k** are said to form a right handed triad or the **fundamental triad.**

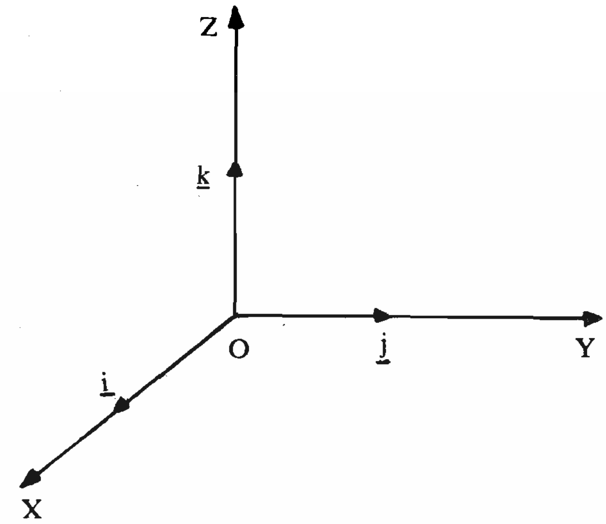


Fig. 12 : Rectangular coordinate system

Note that in this course we shall always be assuming our coordinate system to be a right-handed system.

We now give an important result in the form of the following theorem:

Theorem 1 : Every vector **v** in space may be written as a linear combination of **i, j, k** that is, there exist numbers x,y,z such that

$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

where **i, j, k** are unit vectors along OX, OY, OZ respectively.

Proof : Let $\overrightarrow{OP}$ represent the vector **v** (Fig. 13). Let the coordinates of P be (x, y, z). Let M be the foot of the perpendicular from P on the XOY plane and let MN be drawn parallel to OX. Then ON = y, NM = x and MP = z.

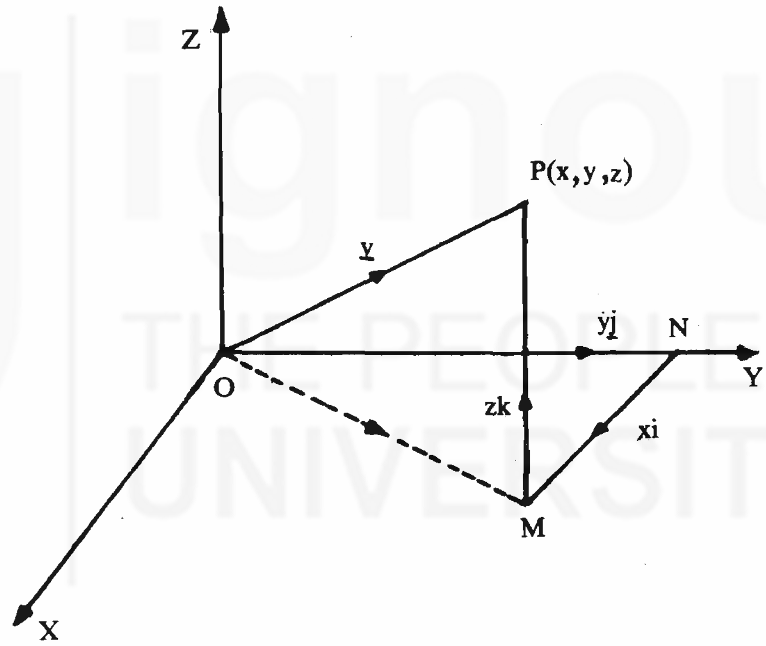


Fig. 13

Since $\overrightarrow{OP} = \mathbf{v}$, we have from ΔOMP,

$\mathbf{v} = \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$

From ΔOMN,

$\overrightarrow{OM} = \overrightarrow{ON} + \overrightarrow{NM}$

Therefore,

$\mathbf{v} = \overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NM} + \overrightarrow{MP}$ .... (3)

Now $\overrightarrow{NM}$ is a vector parallel to OX, we have $\overrightarrow{NM} = x\mathbf{i}$, since the length NM = x units.

Similarly, $\overrightarrow{ON} = y\mathbf{j}$ and $\overrightarrow{MP} = z\mathbf{k}$.

Hence, from (3),

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \ldots. (4)$$

The magnitude of the vector **v** is the length OP and is given by

$$|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$$

Corollary 1 : If P (x_1,y_1,z_1) and Q (x_2,y_2,z_2) are two points in space, then,

$$\overrightarrow{PQ} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

Proof : We have

$$\overrightarrow{OP} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$$

$$\overrightarrow{OQ} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{QP}$$
$$= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

Corollary 2 : If $\mathbf{u} = x_1 i + y_1 j + 2_1 k$

$\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

then $\mathbf{u} + \mathbf{v} = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k}$.

You may verify this yourself.

Example 4 : If $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = -\mathbf{j} + 4\mathbf{k}$, then find $|2\mathbf{u} - \mathbf{v}|$

Solution: $2\mathbf{u} - \mathbf{v} = 2(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) - (-\mathbf{j} + 4\mathbf{k})$

$$= 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \mathbf{j} - 4\mathbf{k}$$
$$= 2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$$

$$|2u - \upsilon| = \sqrt{4 + 25 + 4} = \sqrt{33},$$

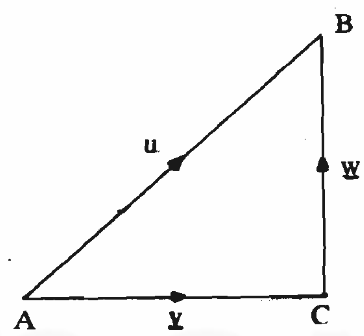


Fig. 14

Example 5 : Show that the vectors

$\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$

form the sides of a right-angled triangle.

Solution : We notice that,

$$\mathbf{v} + \mathbf{w} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$$
$$= 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} = \mathbf{u} \text{ (Fig. 14)}.$$

This shows that the vectors, **u, v, w** form a triangle. Further,

$$|\mathbf{u}| = \sqrt{9 + 16 + 16} = \sqrt{41} = \overrightarrow{AB}$$

$$|\mathbf{v}| = \sqrt{4 + 1 + 1} = \sqrt{6} = \overrightarrow{AC}$$

$$|\mathbf{w}| = \sqrt{1 + 9 + 25} = \sqrt{35} = \overrightarrow{CB}$$

Thus $\overrightarrow{AB}^2 = \overrightarrow{AC}^2 + \overrightarrow{CB}^2$. Hence, the ΔABC is a right-angled triangle.

You will now enjoy doing the following exercises:

E6) If $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$,

$\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$,

$\mathbf{w} = \mathbf{i} - \mathbf{k}$,

Compute $\mathbf{u} + \mathbf{v}$, $(\mathbf{u} - \mathbf{v}) + \mathbf{w}$, $5\mathbf{u} - 3\mathbf{v}$.

E7) If $\mathbf{u} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$

compute a) $|\mathbf{u} + \mathbf{v}|$ b) $|\mathbf{u} - \mathbf{v}|$

E8) Prove that the vectors

$\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

$\mathbf{c} = 4\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$

form the sides of a triangle. Also find the length of the median bisecting vector **c**.

E9) Find a unit vector parallel to the sum of the vectors
$4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $2\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$.

5.3.4 Position Vector

In Theorem 1, we have already shown that if P(x,y,z) is any point in space, and v is a vector such that $\overrightarrow{OP} = \mathbf{v}$, then $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ (ref. Fig. 13). Here x,y,z are called the **components** of the vector v along the coordinate axes. They also represent the projection of the vector v on OX, OY and OZ respectively. The vector v is then called the **position vector** of P relative to O as origin.

Let us now obtain the position vector of a point dividing a given line in the given ratio.

Theorem 2 : If the position vectors of two points A and B referred to O as origin are **u** and v respectively, the position vector of the point dividing AB in the ratio m : n is

$$\mathbf{w} = \frac{n\mathbf{u} + m\mathbf{v}}{n + m}$$

Proof : Let the coordinates of A and B be (x_1,y_1,z_1) and (x_2,y_2,z_2) respectively. (Fig. 15)

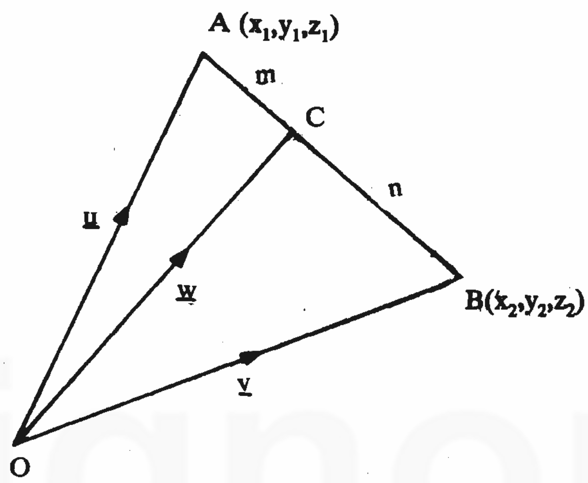


Fig. 15

The coordinates of the point C dividing AB in the ratio m : n are

$$C\left[\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right]$$

Since **u** is the position vector or $A(x_1,y_1,z_1)$

$$\mathbf{u} = \overrightarrow{OA} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \qquad \ldots. \quad (5)$$

$$\text{Similarly, } \mathbf{v} = \overrightarrow{OB} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k} \qquad \ldots. \quad (6)$$

Let **w** be the position vector of C. Then,

$$\mathbf{w} = \overrightarrow{OC} = \left[\frac{x_2m + x_1n}{m + n}\mathbf{i} + \frac{y_2m + y_1n}{m + n}\mathbf{j} + \frac{z_2m + z_1n}{m + n}\mathbf{k}\right]$$

$$= \frac{n}{m + n}(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) + \frac{n}{m + n}(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})$$

$\therefore$ using (5) and (6) we get

$$\mathbf{w} = \frac{m\mathbf{v} + n\mathbf{u}}{m + n}$$

Note that if C is the mid point of AB then m = n. Let m = n = 1, then

$$\mathbf{w} = \frac{1}{2}(\mathbf{u} + \mathbf{v}) \text{ or } 2\ \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$$

Example 6 : The position vectors of P, Q, R, S are $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $2\mathbf{i} + 5\mathbf{j}$, $3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{i} - 6\mathbf{j} - \mathbf{k}$, respectively. Prove that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{RS}$.

Solution : Let O be the origin (see Fig. 16). We have,

$\vec{OP} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\vec{OQ} = 2\mathbf{i} + 5\mathbf{j}$, $\vec{OR} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\vec{OS} = \mathbf{i} - 6\mathbf{j} - \mathbf{k}$

In ΔOPQ, $\vec{OP} + \vec{PQ} = \vec{OQ}$

or $\vec{PQ} = \vec{OQ} - \vec{OP} = 2\mathbf{i} + 5\mathbf{j} - (\mathbf{i} + \mathbf{j} + \mathbf{k})$
$= \mathbf{i} + 4\mathbf{j} - \mathbf{k}$

In ΔORS, $\vec{OR} + \vec{RS} = \vec{OS}$

or $\vec{RS} = \vec{OS} - \vec{OR} = \mathbf{i} - 6\mathbf{j} - \mathbf{k} - (3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$
$= -2\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$
$= -2(\mathbf{i} + 4\mathbf{j} - \mathbf{k})$
$= -2\vec{PQ}$

This shows that $\vec{PQ}$ is parallel to $\vec{RS}$.

You may now try this exercise.

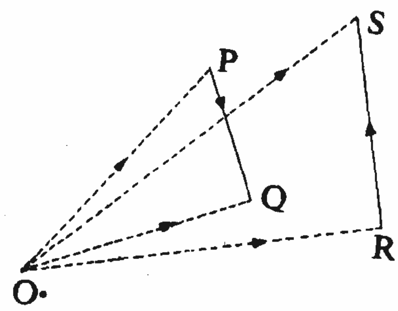


Fig. 16

E10) Show that the points represented by $\mathbf{b} - 2\mathbf{c}$, $\mathbf{a} + 3\mathbf{b} - 3\mathbf{c}$, $2\mathbf{a} + 5\mathbf{b} - 4\mathbf{c}$ are collinear.

The multiplication of a vector by a scalar has already been defined in Section 5.2. When we multiply a vector by another vector, we must define precisely what we mean. There are, in general, two different ways in which vectors can be multiplied. These are the **scalar** and **vector** products of two vectors and are discussed below.

5.3.5 Scalar or Dot Product

The scalar product of two vectors **u** and **v** is defined to be the product of their magnitudes and the cosine of the angle between them (ref. Fig. 17). This is written as

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta \text{ (when } \mathbf{u} \neq 0, \mathbf{v} \neq 0) \qquad \ldots. (7)$$

$\mathbf{u} \cdot \mathbf{v} = 0$ (when u = 0 or = 0)

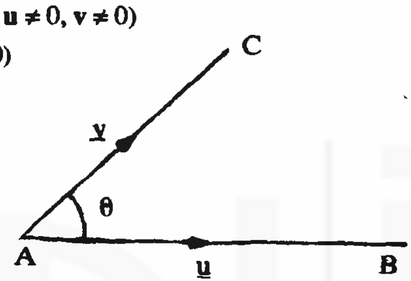


Fig. 17 : Angle between vectors

Because of the notation, this is read as **a** dot **b** and is also called the **dot product.** Here θ ($0 \leq \theta \leq \pi$) is the angle between **u** and **v** (computed when the vectors have their initial point coinciding). Since, $-1 \leq \cos\theta \leq 1$, the dot product **u**.**v** has maximum value when $\cos\theta = 1$. That is, when the vectors **u** and **v** are parallel.

The value of the dot product is a scalar (a real number) and this motivates the term "scalar product". The dot product will be positive, zero or negative according as the angle θ between the vectors is acute, right or obtuse. Let us now consider some important properties of the scalar product.

P 1 : From the definition of a scalar product it follows that

$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

that is, scalar product is **commutative** for every pair of vectors **u**, **v**.

P 2 : We shall now prove that scalar product is also **distributive** with respect to the addition of vectors, that is

$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

Referring to Fig. 18, we have

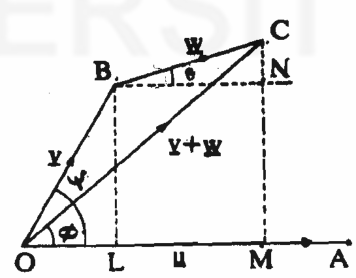


Fig. 18 : Distributive law for scalar product

$$\mathbf{u}.(\mathbf{v} + \mathbf{w}) = |\mathbf{u}|\,|\mathbf{v} + \mathbf{w}|\cos\phi = |\mathbf{u}|\,OC\cos\phi = |\mathbf{u}|\,OM \qquad \ldots. (8)$$

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\psi = |\mathbf{u}|\,OB\cos\psi = |\mathbf{u}|\,OL \qquad \ldots. (9)$$

$$\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}|\,|\mathbf{w}|\cos\theta = |\mathbf{u}|\,BC\cos\theta = |\mathbf{u}|\,BN = |\mathbf{u}|\,LM \qquad \ldots. (10)$$

Adding (9) and (10) we get,

$\mathbf{u}.\mathbf{v} + \mathbf{u}.\mathbf{w} = |\mathbf{u}|(OL + LM) = |\mathbf{u}|\,OM = \mathbf{u}.(\mathbf{v} + \mathbf{w})$ from (8)

P 3 : From the definition of dot product it is clear that $(m\mathbf{u}.n\mathbf{v}) = mn(\mathbf{u}.\mathbf{v})$, where m and n are scalars.

P 4 : The scalar product of two non-zero vectors is zero if the vectors are at right angles:

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = 0$ **when** $\theta = \pi/2$

P 5 : $\mathbf{u}.\mathbf{u} = |\mathbf{u}||\mathbf{u}|\cos\theta = |\mathbf{u}|^2$ or $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}\,(\geq 0)$

The scalar product of a vector with itself is equal to the square of the magnitude of the vector. From this and (7) follows the useful formula $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\sqrt{\mathbf{u}\cdot\mathbf{u}}\sqrt{\mathbf{v}\cdot\mathbf{v}}}$.

P 6 : Since **i, j, k** are unit vectors at right angles to each other it follows on using P1 and P4, that

$\mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{j} = 0, \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0, \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$

and use of P5 yields

$\mathbf{i}\cdot\mathbf{i} = 1;\ \mathbf{j}\cdot\mathbf{j} = 1, \mathbf{k}\cdot\mathbf{k} = 1$

P 7 : If the vectors **u** and v are written interms of their components as

$\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$

$\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k},$

then, using P2, P3 and P5 we have,

$$\begin{aligned}\mathbf{u}.\mathbf{v} &= (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}).(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) \\ &= x_1x_2\mathbf{i}.\mathbf{i} + x_1y_2\mathbf{i}.\mathbf{j} + x_1z_2\mathbf{i}.\mathbf{k} + y_1x_2\mathbf{j}.\mathbf{i} + y_1y_2\mathbf{j}.\mathbf{j} + y_1z_2\mathbf{j}.\mathbf{k} + z_1x_2\mathbf{k}.\mathbf{i} + z_1y_2\mathbf{k}.\mathbf{j} + z_1z_2\mathbf{k}.\mathbf{k} \\ &= x_1x_2 + y_1y_2 + z_1z_2\end{aligned}$$

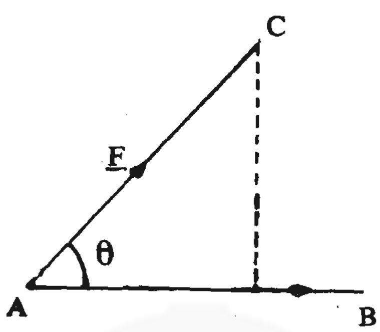


Fig. 19

Now suppose that **u** and v (=0) are arbitrary vectors and ..t is the angle between them. Then the real number $|\mathbf{v}|\cos\theta$ is called the **component of v in the direction of u** or the **projection of v in the direction of u.** Several examples are now considered to illustrate the usefulness of dot product.

Example 7 : If a constant force **F** acts on the particle which moves along a line segment from A to B, then the work done by **F** in the displacement is defined as the product of AB and the component of F in the direction of AB, that is,

$W\ |\overrightarrow{AB}|\ |\mathbf{F}|\cos\theta = \mathbf{F}\cdot\overrightarrow{AB}$ (see Fig. 19).

Thus, the work done is the scalar product of the force component in the direction of motion and the displacement.

Example 8 : The projection of $12\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ along $2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ is given by

$$\frac{(12\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})\cdot(2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})}{\sqrt{4+16+16}}$$

$$= \frac{24 - 12 + 24}{6} = 6.$$

Example 9 : Find the work done by the force $\mathbf{F} = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ when its point of application moves from (A(1, −2, −2) to B(3,1,1).

Solution : Let O be the origin.

Then, $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $\overrightarrow{OB} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = (3-1)\mathbf{i} + (1+2)\mathbf{j} + (1+2)\mathbf{k} \\ &= 2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}\end{aligned}$$

$$\begin{aligned}\text{Work done} &= \mathbf{F}.\mathbf{AB} \text{ (ref. Example 7)} = (5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})\cdot(2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}) \\ &= 10 - 6 + 9 = 13 \text{ units.}\end{aligned}$$

Example 10 : If $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$, then determine the angle between u and v.

Solution : $\mathbf{u}.\mathbf{v} = 2 \cdot 3 + 3 + (-1)\cdot(1) + 4 \cdot 1 = 6 - 1 + 4 = 9$

If θ is the angle between the vectors, then

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= |\mathbf{u}||\mathbf{v}|\cos\theta = \sqrt{4+1+16}\ \sqrt{9+1+1}\cos\theta \\ &= \sqrt{21}\sqrt{11}\cos\theta\end{aligned}$$

Equating the two values of **u.v**,

$9 = \sqrt{21}\sqrt{11}\cos\theta.$

Hence, $\theta = \cos^{-1}\frac{9}{\sqrt{21}\sqrt{11}}$

Example 11 : Determine p so that the vectors **2i**+ p**j** = **k** and **4i** + **2j** − **2k** are at right angles.

Solution : If the vectors are at right angles, their dot product must be zero. That is,

(2**i** + p**j** + **k**). (4**i** − 2**j** − 2**k**)= 0

or 2.4 + p.(−2) + 1.(−2) = 0

or 8 −2p − 2 = 0. or p = 3.

Example 12: Prove that in a right-angled triangle ABC, right angled at B,

$AC^2 = AB^2 + BC^2$

Solution : Let $\overrightarrow{AB} = \mathbf{u}$, and $\overrightarrow{BC} = \mathbf{v}$ (Fig. 20)

Then, by P3, **u**.**v** = 0.

In ΔABC, $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$.

Using P4, we have

$$\begin{aligned} AC^2 = |\overrightarrow{AC}|^2 = \overrightarrow{AC}\cdot\overrightarrow{AC} &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} + (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} \\ &= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2, \text{ since } \mathbf{u}\cdot\mathbf{v} = 0 \\ &= AB^2 + BC^2 \end{aligned}$$

Hence, the required result.

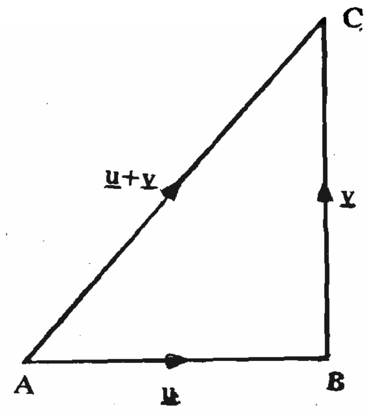


Fig. 20

You may now try these exercises.

E11) If **a** = 4**i** − 2**j** + 4**k**, **b** = 3**i** − 6**j** − 2**k**, find **a**.**b** and the angle between **a** and **b**.

E12) If **a** = **i** + 2**j** + 3**k**, **b** = −**i** + 2**j** + **k**, **c** = 3**i** + **j** find p such that **a** + p**b** is perpendicular to **c**.

E13) Show that the vectors **u** = 3**i** − 2**j** + **k**, **v** = **i** − 3**j** + 5k, and **w** = 2**i** + **j** − 4**k** form a right-angled triangle.

E14) Prove that the sum of the squares of diagonals of a parallelogram is equal to the sum of the squares of its sides.

E15) Three vectors **u**,**v**,**w** are of magnitudes 3, 4, 5 respectively. **u** is perpendicular to **v** +w; **v** is perpendicular to **u**+w and **w** is perpendicular to **u** + v. Show that $|\mathbf{u} + \mathbf{v} + \mathbf{w}| = 5\sqrt{2}$.

5.3.6 Vector or Cross Product

We have seen that the scalar product of two vectors is a scalar quantity which is maximum when the two vectors are parallel, and zero when they are perpendicular to each other. The vector product of two vectors, on the other hand, is a vector quantity which is maximum when the two vectors are perpendicular to each other and zero when they are parallel. We now define it as follows.

If **u** and **v** are the two vectors then their **vector product** is written as

$\mathbf{u} \times \mathbf{v}$

and is a vector. Let **w** = **u** × **v**, then **w** is defined as follows:

i) If **u** and **v** have the same or opposite direction or one of these vectors is the zero vector, then **w** = 0.

ii) In any other case, **w** is the vector whose length is equal to the area of the parallelogram with **u** and **v** as adjacent sides, and whose direction is perpendicular to both **u** and **v**. **u**.**v**,**w** in this order, form a right handed triad, as shown in Fig. 21(a).

Since the parallelogram with **u** and **v** as adjacent sides has the area $|\mathbf{u}|\ |\mathbf{v}| \sin\theta$, the vector product of the vectors **u** and **v** making an angle θ, is given by

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = (|\mathbf{u}|\ |\mathbf{v}| \sin\theta)\mathbf{n}$$

when **n** is a unit vector prependicular to the plane **u** and **v**, the positive direction of n being determined by the right-handed screw convention; that is, if we place a screw at 0

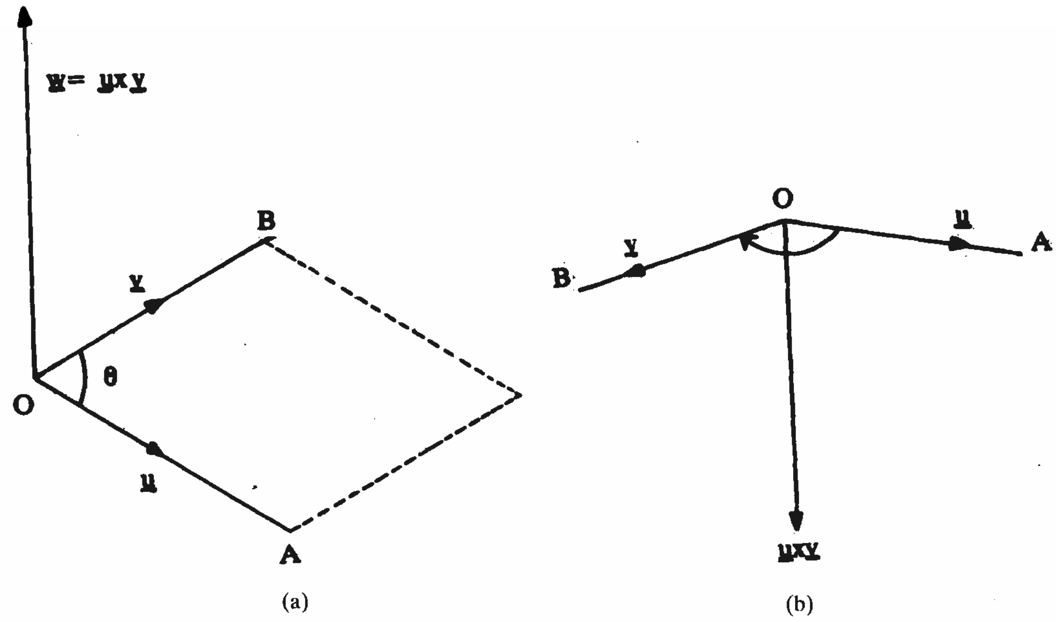


Fig. 21 : Vector product of two vectors

perpendicular to the plane of **u** and **v**, and rotate it from **u** to **v**, the direction in which the screw moves is the positive direction of **n**.
In Fig. 21(a) the positive direction of **n** is perpendicular to the plane of **u** and **v** and directed out of it while in Fig. 21(b) the positive direction of **n** is perpendicular to the plane of **u** and **v** and going into it.

Note that the vectors, **u, v, w** in Fig. 21(a) which are in the positive direction of the axes form a **right-handed** triad, whereas **u, v, w** in Fig. 21(b) form a **left-handed triad.**
vector product **u** × **v**, is also called the cross product because of the notations used and is read as "**u** cross **v**".
The magnitude of **u** × **v** is given by

$$|\mathbf{w}| = |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta\,|\mathbf{n}|$$

$$= |\mathbf{u}|\,|\mathbf{v}| \sin\theta, \text{ since } |\mathbf{n}| = 1$$

As $0 \le \theta \le 180^0$, sin θ cannot be negative so that $|\mathbf{w}|$ will not be negative. Also, since the maximum value of sin θ = 1, $|\mathbf{w}|$ is maximum when **u** and **v** are perpendicular. Let us now see some properties of the cross product.
PC 1 : Clearly if rotation from **u** to **v** moves the screw upwards then rotation from **v** to **u** will move it downwards. Thus

$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$;

that is, cross product is not commutative. The order of the factors in a vector product is, therefore, of great importance and must be carefully observed.
PC 2 : The vector product of two vectors is zero if the vectors are parallel. That is,

$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}|\,|\mathbf{v}| \sin 0)\mathbf{n} = 0$

In particular, $\mathbf{u} \times \mathbf{u} = 0$.
Also, by definition, the vector product of any vector with the zero vector is the zero vector, that is, $\mathbf{u} \times \mathbf{0} = \mathbf{0}$.
PC 3 : If m and n are scalars then $(m\mathbf{u} \times n\mathbf{v}) = mn(\mathbf{u} \times \mathbf{v})$
PC 4 : For the unit vectors **i, j, k** introduced earlier, we have

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$

$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$

$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

where **i, j, k** in this order form a right-handed triad.

PC 5 : If $\mathbf{u} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$,

$\mathbf{v} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

then using PC3 and PC4, the vector product of **u** and **v** can be expressed as follows:

$$\mathbf{u}\times\mathbf{v} = (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \times (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})$$
$$= x_1x_2\mathbf{i}\times\mathbf{i} + x_1y_2\mathbf{i}\times\mathbf{j} + x_1z_2\mathbf{i}\times\mathbf{k} + y_1x_2\mathbf{j}\times\mathbf{i} + y_1y_2\mathbf{j}\times\mathbf{j} + y_1z_2\mathbf{j}\times\mathbf{k}$$
$$+ z_1x_2\mathbf{k}\times\mathbf{i} + z_1y_2\mathbf{k}\times\mathbf{j} + z_1z_2\mathbf{k}\times\mathbf{k}$$
$$= x_1y_2\mathbf{k} - x_1z_2\mathbf{j} - y_1x_2\mathbf{k} + y_1z_2\mathbf{i} + z_1x_2\mathbf{j} - z_1y_2\mathbf{i}$$
$$= (y_1z_2 - y_2z_1)\mathbf{i} + (z_1x_2 - z_2x_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}$$

A convenient way to express the vector product is to write it as a determinant, namely.

$$\mathbf{u}\times\mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$$

PC 6 : Vector product is distributive with respect to vector addition, that is,

$\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$

E16) Prove PC 6 by taking **u,v,w** in the component form and making use of PC 3, PC 4 and PC 5.

We now give a simple geometrical application of the vector product, and illustrates it with examples.

The magnitude of the vector product u × v is equal to twice the area of the triangle u and v as two of its sides.

Clearly, from Fig. 22, area of $\Delta OAB = (1/2)\, OA \cdot OB \sin\theta$

$$= (1/2)\ |\mathbf{u}|\ |\mathbf{v}|\sin\theta$$
$$= (1/2)\ |\mathbf{u}\times\mathbf{v}|$$

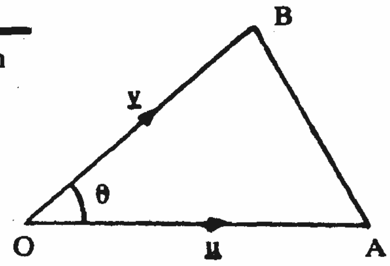


Fig. 22

Example 3 : If $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, then find $|\mathbf{u}\times\mathbf{v}|$ and a unit vector perpendicular to both **u** and **v**.

Solution : Using PC 5,

$\mathbf{u}\times\mathbf{v} = (y_1z_2 - y_2z_1)\mathbf{i} + (z_1x_2 - z_2x_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}$

where $(x_1, y_1, z_1) = (2, 2, -1)$ and $(x_2, y_2, z_2) = (1, -3, 2)$ thus,

$\mathbf{u}\times\mathbf{v} = \mathbf{i}(4-3) - \mathbf{j}(4+6) + \mathbf{k}(-6-12)$

$= \mathbf{i} - 10\mathbf{j} - 18\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ (say), so that

$|\mathbf{u}\times\mathbf{v}| = \sqrt{x^2+y^2+z^2}$

or $|\mathbf{u}\times\mathbf{v}| = \sqrt{1+100+324} = \sqrt{425}$ (since $x = 1$)

$y = -1, z = -18.$

Since **u** × **v** is perpendicular to the plane containing **u** and **v**, it is perpendicular to both **u** and **v**. Hence $\dfrac{\mathbf{u}\times\mathbf{v}}{|\mathbf{u}\times\mathbf{v}|}$ is a unit vector perpendicular to **u** and **v**. Thus,

$$\frac{\mathbf{i} - 10\mathbf{j} - 18\mathbf{k}}{\sqrt{425}}$$

is perpendicular to **u** and **v**.

How about doing some exercises before going to the next example?

E17) If $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, compute $\mathbf{a}\times\mathbf{b}$ and $|\mathbf{a}\times\mathbf{b}|$.

E18) If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, compute $\mathbf{a}\times\mathbf{b}$ and $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})$.

E19) If $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find a unit vector perpendicular to **a** and **b**.

E20) Find the area of the triangle whose vertices are A (1, 2, 3), B(2, 5, −1), C(−1, 1, 2).

Example 14: Find the area of the parallelogram where diagonals are given by $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

Solution : Area of the parallelogram ABCD (see Fig. 23) is

$= 4$ (Area of Δ AOB)

$= 4(1/2)\ |\overrightarrow{AO}\times\overrightarrow{BO}| = 2\ |\overrightarrow{AO}\times\overrightarrow{BO}|$

$= 2\ |1/2(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})\times 1/2(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}|$

Now, $(3\mathbf{i}+\mathbf{j}-2\mathbf{k})\times(\mathbf{i}-3\mathbf{j}+4\mathbf{k}) = (-2\mathbf{i}-14\mathbf{j}-10\mathbf{k})$ verify it!)

Hence area $= \dfrac{1}{2}|-2\mathrm{i} - 14\mathrm{j} - 10\mathrm{k}| = \dfrac{1}{2}\sqrt{4 + 196 + 100} = 5\sqrt{3}.$

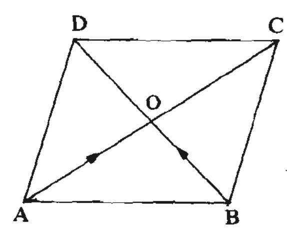


Fig. 23

You may now try the following exercises:

E21) Find the product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ where

$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{c} = -3\mathbf{i} + 5\mathbf{j}$.

E22) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are any three vectors prove that

$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) + \mathbf{v} \times (\mathbf{w} + \mathbf{u}) + \mathbf{w} \times (\mathbf{u} + \mathbf{v}) = 0$

E23) Prove that

$(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v}) = 2(\mathbf{u} \times \mathbf{v})$.

5.4 SUMMARY

In this unit we have studied the following:

1) A quantity having magnitude and direction is called a vector.
2) Vectors can be represented by directed line segments.
3) Vectors are added by the triangle law and the parallelogram law.
4) A vector can be expressed in the component form as
   $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,
   where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along OX, OY, OZ, respectively.
5) The scalar product of two vectors is defined as
   $\mathbf{u}.\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$
6) Various properties of dot product.
7) The vector product of two vectors is defined as
   $\mathbf{u} \times \mathbf{v} = (|\mathbf{u}|\,|\mathbf{v}|\sin\theta)\,\mathbf{n}$,
8) Properties of cross product.

5.5 SOLUTIONS/ANSWERS

E1) a) (1,1,1)

b) (0,0,4)

c) (0,0,0)

d) (2,2,1)

E2) a) In Fig. 24, $\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$ and

$$\frac{\overrightarrow{BC}}{\overrightarrow{AD}} = \frac{4}{7} \text{ or } \overrightarrow{BC} = \frac{4}{7}\overrightarrow{AD}$$

Hence, $\overrightarrow{BC} = \frac{4}{7}\overrightarrow{AD} = \frac{4}{7} \cdot 7\mathbf{u} = 4\mathbf{u}$

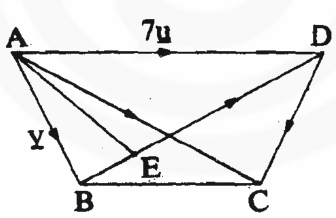


Fig. 24

b) In Δ ABC, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ or $\mathbf{v} + 4\mathbf{u} = \overrightarrow{AC}$

c) $\overrightarrow{BD} = 7\mathbf{u} - \mathbf{v}$

d) $\overrightarrow{DC} = \mathbf{v} - 3\mathbf{u}$

e) $\overrightarrow{AE} = \frac{1}{11}(7\mathbf{v} + 28\mathbf{u})$

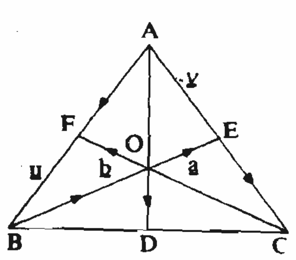


Fig. 25

E3) Let $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{AC} = \mathbf{v}$ (Fig. 25)

a) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ or $\overrightarrow{BC} = \mathbf{v} - \mathbf{u}$

In Δ ABD, $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ or $\mathbf{u} + \frac{1}{2}(\mathbf{v} - \mathbf{u}) = \overrightarrow{AD}$

or $\overrightarrow{AD} = \frac{1}{2}(\mathbf{u} + \mathbf{v})$.

In Δ ABE, $\overrightarrow{AB} + \overrightarrow{BE} = \overrightarrow{AE}$, or $\overrightarrow{BE} = \frac{1}{2}(\mathbf{v} - \mathbf{u})$

In Δ ACF, $\overrightarrow{AC} + \overrightarrow{CF} = \overrightarrow{AF}$, or $\overrightarrow{CF} = \frac{1}{2}(\mathbf{u} - \mathbf{v})$

b) Let $\overrightarrow{BE} = \mathbf{a}$, $\overrightarrow{CF} = \mathbf{b}$.

Let O be the point of intersection of medians. The medians are divided at the point O in the ratio 2:1. Therefore,

$\overrightarrow{BO} = \frac{2}{3}\overrightarrow{BE} = \frac{2}{3}\mathbf{a}$

$\overrightarrow{CO} = \frac{2}{3}\overrightarrow{CF} = \frac{2}{3}\mathbf{b}$

In Δ BCO, $\overrightarrow{BC} + \overrightarrow{CO} = \overrightarrow{BO}$ or $\overrightarrow{BC} = \frac{2}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$

Similarly, $\overrightarrow{CA} = \frac{2}{3}\mathbf{a} + \frac{4}{3}\mathbf{b}$, $\overrightarrow{AB} = -\frac{4}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$

$\overrightarrow{AD} = -\mathbf{a} - \mathbf{b}$.

c) $\overrightarrow{AC} = 2(\overrightarrow{AB} + \overrightarrow{BE})$ (Fig. 25)

$\overrightarrow{BC} = \overrightarrow{AB} + 2\overrightarrow{BE}$

$\overrightarrow{AD} = \frac{3}{2}\overrightarrow{AB} + \overrightarrow{BE}$

$\overrightarrow{CF} = -\frac{3}{2}\overrightarrow{AB} - 2\overrightarrow{BE}$

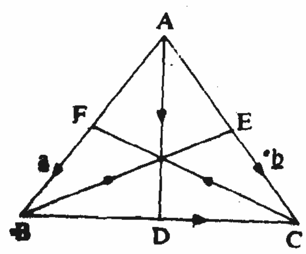


Fig. 26

d) Let $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AC} = \mathbf{b}$ (see Fig. 26). Then $\overrightarrow{BC} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{BD} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$ and in Δ ABD we have $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ or $\overrightarrow{AD} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$

In Δ BEA, $\overrightarrow{AB} + \overrightarrow{BE} = \overrightarrow{AE}$ or $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{AC} - \overrightarrow{AB} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

In Δ BCF, $\overrightarrow{BC} + \overrightarrow{CF} = \overrightarrow{BF} = -\overrightarrow{FB}$ or $\overrightarrow{CF} = \frac{1}{2}\mathbf{a} - (\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} - \mathbf{b})$

Hence $\overrightarrow{AD} = \overrightarrow{BE} + \overrightarrow{CF} = \mathbf{a} + \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} - \mathbf{a} + \frac{1}{2}\mathbf{a} - \mathbf{b} = 0$.

E4) a) Let $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{AD} = \mathbf{v}$. Then $\overrightarrow{BC} = \mathbf{v}$ (see Fig. 27)

In ΔABC, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ or $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$

In ΔABD, $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ or $\overrightarrow{BD} = \mathbf{v} - \mathbf{u}$

b) $\overrightarrow{AB} = \frac{1}{2}(\overrightarrow{AC} - \overrightarrow{BD})$, $\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{BD})$

c) $\overrightarrow{AB} = \overrightarrow{AD} - \overrightarrow{BD}$, $\overrightarrow{AC} = 2\overrightarrow{AD} - \overrightarrow{BD}$.

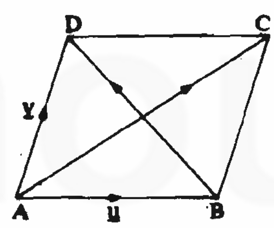


Fig. 27

E5) Let E be the mid point of CD. Let $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{AD} = \mathbf{v}$ (Fig. 28)

Then, $\overrightarrow{DE} = \frac{1}{2}\mathbf{u}$

In Δ AED, $\overrightarrow{AD} + \overrightarrow{DE} = \overrightarrow{AE}$ or $\overrightarrow{AE} = \mathbf{v} + \frac{1}{2}\mathbf{u}$.

Now AO is a part of AE. Let $\overrightarrow{AO} = m\left(\mathbf{v} + \frac{1}{2}\mathbf{u}\right)$

In Δ ABD, $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$ or $\overrightarrow{BD} = \mathbf{v} - \mathbf{u}$.

The vector $\overrightarrow{BO}$ is a part of $\overrightarrow{BD}$. Let $\overrightarrow{BO} = n(\mathbf{v} - \mathbf{u})$.

In Δ ABD, $\overrightarrow{AB} + \overrightarrow{BO} = \overrightarrow{AO}$ or $\mathbf{u}\left(1 - n - \frac{m}{2}\right) + \mathbf{v}(n - m) = 0$.

Since u and v are non-parallel vectors, we must have

$n = m = \frac{2}{3}$. Thus, $\overrightarrow{BO} = \frac{2}{3}\overrightarrow{BD}$.

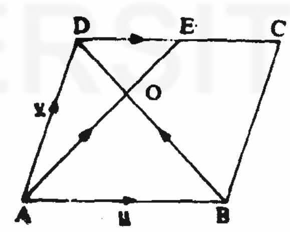


Fig. 28

E6) $\mathbf{u} + \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

$\mathbf{u} + \mathbf{v} + \mathbf{w} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$.

$5\mathbf{u} - 3\mathbf{v} = 2\mathbf{i} + 7\mathbf{j} + 12\mathbf{k}$.

E7) a) $|\mathbf{u} - \mathbf{v}| - 9$

b) $|\mathbf{u} - \mathbf{v}| = \sqrt{33}$

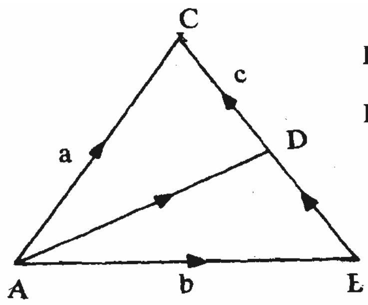


Fig. 29

E8) We notice that $\mathbf{b}+\mathbf{c}=\mathbf{a}$. Thus the given vectors form the sides of a triangle (Fig. 29).

In Δ ABD, $\vec{AB}+\vec{BD}=\vec{AD}$ or $\vec{AD}=\mathbf{b}+\frac{1}{2}\mathbf{c}$ and $|\vec{AD}|=\sqrt{6}$

E9) $\frac{1}{9}(6\mathbf{i}-6\mathbf{j}-3\mathbf{k})$

E10) Let A, B, C be three points.

Then $\vec{OA}=\mathbf{b}-2\mathbf{c}$, $\vec{OB}=\mathbf{a}+3\mathbf{b}-3\mathbf{c}$, $\vec{OC}=2\mathbf{a}+5\mathbf{b}-4\mathbf{c}$

$\vec{AB}=\vec{OB}-\vec{OA}=\mathbf{a}+2\mathbf{b}-\mathbf{c}$

$\vec{BC}=\vec{OC}-\vec{OB}=\mathbf{a}+2\mathbf{b}-\mathbf{c}$

Thus, $\vec{AB}$ and $\vec{BC}$ are parallel. But they have a common point B. Hence A,B,C are collinear.

E11) $\mathbf{a}\cdot\mathbf{b}=16$ and $\cos\theta=\frac{16}{6\times 7}=\frac{8}{21}$

E12) If $\mathbf{a}+p\mathbf{b}$ is perpendicular to $\mathbf{c}$.

$(\mathbf{a}+p\mathbf{b}).\mathbf{c}$ which gives $p=5$.

E13) Notice that $\mathbf{w}+\mathbf{v}=\mathbf{u}$. This means that the vectors form a triangle. Verify that $\mathbf{u}.\mathbf{w}=0$. Thus $\mathbf{u}$ and $\mathbf{w}$ are at right angles.

E14) Let $\vec{AB}=\mathbf{u}$, $\vec{AD}=\mathbf{v}$. (see Fig. 27). Then $\vec{AC}=\mathbf{u}+\mathbf{v}$, $\vec{BD}=\mathbf{v}-\mathbf{u}$,

$\vec{AC}^2=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})=\mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}$

$\vec{BD}^2=(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=\mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}$

$$\vec{AC}^2+\vec{BD}^2=2\mathbf{u}\cdot\mathbf{u}+2\mathbf{v}\cdot\mathbf{v}=2|\mathbf{u}|^2+2|\mathbf{v}|^2=2(\vec{AB}^2+\vec{AD}^2)$$
$$=\vec{AB}^2+\vec{DC}^2+\vec{AD}^2+\vec{BC}^2,$$

since $\vec{AB}=\vec{DC}$ and $\vec{AD}=\vec{BC}$

E15) $\mathbf{u}$ is perpendicular to $\mathbf{v}+\mathbf{w}$, therefore $\mathbf{u}.(\mathbf{v}+\mathbf{w})=0$.

Similarly, $\mathbf{v}.(\mathbf{u}+\mathbf{w})=0$ and $\mathbf{w}.(\mathbf{u}+\mathbf{v})=0$

$$\begin{aligned}|\mathbf{u}+\mathbf{v}+\mathbf{w}|^2 &= (\mathbf{u}+\mathbf{v}+\mathbf{w})\cdot(\mathbf{u}+\mathbf{v}+\mathbf{w})\\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})+\mathbf{v}\cdot(\mathbf{u}+\mathbf{w})+\mathbf{v}\cdot\mathbf{v}+\mathbf{w}\cdot(\mathbf{u}+\mathbf{v})+\mathbf{w}\cdot\mathbf{w}\\ &= |\mathbf{u}|^2+|\mathbf{v}|^2+|\mathbf{w}|^2\\ &= 9+16+25=50\end{aligned}$$

Thus, $|\mathbf{u}+\mathbf{v}+\mathbf{w}|^2=5\sqrt{2}$

E17) $\mathbf{a}\times\mathbf{b}=7\mathbf{i}+2\mathbf{j}-18\mathbf{k}$, $|\mathbf{a}\times\mathbf{b}|=\sqrt{377}$

E18) $\mathbf{a}\times\mathbf{b}=10\mathbf{i}+3\mathbf{j}+11\mathbf{k}$.

$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=-20\mathbf{i}-6\mathbf{j}-22\mathbf{k}$.

E19) $\mathbf{a}\times\mathbf{b}=7\mathbf{i}-6\mathbf{j}-10\mathbf{k}$

$|\mathbf{a}\times\mathbf{b}|=\sqrt{49+36+100}=\sqrt{185}$

A unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ is

$$\frac{7\mathbf{i}-6\mathbf{j}-10\mathbf{k}}{\sqrt{185}}$$

E20) $\vec{OA}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

$\vec{OB}=2\mathbf{i}+5\mathbf{j}-\mathbf{k}$

$\vec{OC}=-\mathbf{i}+\mathbf{j}+2\mathbf{k}$

$\vec{AB}=\vec{OB}-\vec{OA}=\mathbf{i}+3\mathbf{j}-4\mathbf{k}$

$\vec{AC}=\vec{OC}-\vec{OA}=-2\mathbf{i}-\mathbf{j}-\mathbf{k}$

Area of Δ ABC $=\frac{1}{2}|\vec{AB}\times\vec{AC}|=\frac{1}{2}\sqrt{155}$

E21) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=-21\mathbf{i}+23\mathbf{j}+27\mathbf{k}$.

E22) L.H.S. $= \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} + \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u} + \mathbf{w} \times \mathbf{v}$
$= \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} - \mathbf{u} \times \mathbf{v} - \mathbf{u} \times \mathbf{w} - \mathbf{v} \times \mathbf{w}$
$= 0.$

E23) $(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v})$
$= \mathbf{u} \times \mathbf{u} + \mathbf{u} \times \mathbf{v} - \mathbf{v} \times \mathbf{u} - \mathbf{v} \times \mathbf{v}$
$= 0 + \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{v} + 0$
$= 2(\mathbf{u} \times \mathbf{v}).$

NOTES