UNIT 3 ELEMENTARY ALGEBRA

Structure

3.1 INTRODUCTION

In Unit 2 we discussed the power function x^n and its graph. x^n is an example of a polynomial. In this unit we will first discuss various polynomials and polynomial equations. In particular, we will discuss algebraic and graphical solutions of linear and quadratic equations.

We then go on to discuss various arrangements of numbers, as well as their sum, that is, sequences and series. We look at arithmetic sequences and series and geometric sequences and series in some detail.

Counting is an old activity of human beings. Sometimes actual counting is not possible or it is highly time-consuming. So, one has to devise ways of counting without actually performing the process of counting. The theory of combinations and permutations is aimed at achieving this objective. We discuss this in Section 3.4.

We end this unit with the binomial theorem.

Objectives

After reading this unit, you should be able to

- define polynomials and perform arithmetic operations on them;
- solve linear equations algebraically and graphically;
- obtain the roots of a quadratic equation;
- identify arithmetic and geometric progressions;
- define arithmetic and geometric series, find their sum upto n terms and upto infinity, wherever possible;
- evaluate the number of permutations and combinations of n objects taken r at a time, where $n > r$;
- state and use the binomial theorem.

3.2 POLYNOMIALS AND EQUATIONS

In previous units you have been exposed to various combinations of variables and constants. In this section we start with discussing algebraic expressions called polynomials.

3.2.1 Polynomials

Consider the following combinations of letters and numbers, linked together by multiplication: 0, 1, x, $(-2)a$, $3ax$, $(3/2)\ axb$, $(-5)x$, $(-3)y$ (where a and b are constants, and x and y are variables). These are examples of **monomials**.

Now consider the expression $2 + 3ax - 3by^2$, where x and y are variables and a and b are constants. It involves three monomials, linked together by the operation of addition. Such an expression is called a polynomial. In general, a monomial, or the sum of monomials, is a **polynomial**. Each monomial that occurs in a polynomial is called a **term** of the polynomial. If a polynomial has two terms we call it a **binomial**.

Can you do the following exercise now?

E 1) Which of the following expressions are polynomials?

$0, (9/5)c + 32, x^4 + 4x^3 - 3x^2 - x^4 + 1, (x^2 - y) - (x^2 + y), x/y$

(Here c, x and y are variables.)

Now let us see what the degree of a monomial is. It is the total exponent of all the variables occurring in it. Thus, the degree of πr^2 is 2 (= exponent of r) and the degree of $-2x^2y^3$ is $2 + 3 = 5$. We define the degree of a constant to be zero. The **degree of a polynomial** is the maximum of the degrees of all the terms occurring in it. Thus, the degree of $2x + axy + y^3 - 5$ (x, y being variables and a being a constant) is the maximum of 1, 1 + 1, 3, 0, which is 3.

Try this exercise now.

E 2) What are the degrees of the polynomials in E 1?

Now, what happens if you put $x = 1$ in the polynomial $x^2 - 2x + 1$? The polynomial's value becomes zero. In such a situation we say that 1 is a **root** (or **zero**) of the polynomial $x^2 - 2x + 1$.

Suppose a real number α is a root of a polynomial $p(x)$ in x. Then $(x - \alpha)$ will be a factor of $p(x)$, that is, $(x - \alpha)$ will divide $p(x)$.

Let us consider an example.

Example 1 : Is $\frac{x^3 - 1}{x - 1}$ a polynomial?

Solution : At first, the given expression does not appear to be a polynomial. But if we look a little closer, we find that 1 is a root of $x^3 - 1$. Now,

$$
\begin{array}{l}
(x-1)\ \big)\ x^3 - 1\ \big(\ x^2 + x + 1 \\
\qquad\quad \underline{x^3 - x^2} \\
\qquad\qquad\ \ x^2 - 1 \\
\qquad\qquad\ \ \underline{x^2 - x} \\
\qquad\qquad\quad\ x - 1 \\
\qquad\qquad\quad\ \underline{x - 1} \\
\qquad\qquad\qquad 0
\end{array}
$$

Thus, $\frac{x^3 - 1}{x - 1}$ is the polynomial $x^2 + x + 1$.

[We can check this by actually calculating $(x - 1)(x^2 + x + 1)$. This equals $x(x^2 + x + 1) - (x^2 + x + 1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1$.]

Note : A polynomial of degree n, in one variable, can have n roots, at the most.

E 3) How many roots can $(x^2 - 9)$ have? Does $(x - 2)$ divide $(x^2 - 9)$?

Now that we have recalled some definitions related to polynomials, we go on to discuss linear equations.

3.2.2 Linear Equations

A polynomial of degree one, like $x + y - 1$, is a linear polynomial. When we equate a linear polynomial to zero, we get a linear equation. For example, $x + 5 = 0$ and $x + 2y - 3z = 9$ are linear equations.

The general linear equation in one variable x is ax + b = 0, where a and b are constants and $a \neq 0$. This has only one solution, x = − (b/a).

On the other hand, **a linear equation in two or more variables has infinitely many solutions.**

Look at the following example.

Example 2: Obtain the solutions of the equation 3x − 5 = 2y.

Solution : The given equation is equivalent to the equation 3x − 2y − 5 = 0. For any value x_0, of x, and $y_0 = (3x_0 - 5)/2$ of y, the pair (x_0, y_0) is a solution of the given equation. Thus, the equation has infinitely many solutions. They are the points on the line 3x − 2y − 5 = 0 given in the graph in Fig. 1. This is the graph of the function y = (3x − 5)/2 (see Unit 2).

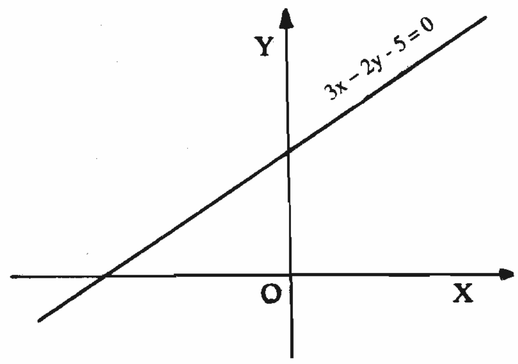

Fig. 1 : 3x − 2y − 5 = 0

In general, a solution of the linear equation ax + by + c = 0, a, b, c ∈ **R** and b ≠ 0, is $\left(x_0, \frac{-(ax_0 + c)}{b}\right)$, for each $x_0 \in$ **R**.

Now you can try the following exercise.

E 4) How many solutions do the following equations have?

a) 2x = 3, b) 5 − 2x = y + 3, c) 7 = 2y.

Give at least one solution for each of them.

You will often come across situations where you need to solve two or more linear equations at a time. For example, you may need to find a pair (x, y) that satisfies the equation 2x = 3 as well as the equation 3x − 2y − 5 = 0. That is, you will be trying to solve a system of simultaneous linear equations. Let us look at the above example in detail.

Example 3 : Solve the simultaneous linear equations

2x = 3 and 3x − 2y − 5 = 0.

Solution : Putting the solution x = 3/2 of the first equation in the second equation, we get

3 (3/2) − 2y − 5 = 0 ⇒ y = − 1/4.

Thus, the system of simultaneous equations has the unique solution (3/2, − 1/4).

A system of simultaneous linear equations may or may not have a solution. If the system of equations has a solution, it is called **consistent**. Otherwise the system is called **inconsistent**. Thus, the system in Example 3 is consistent. Let us look at an example of an inconsistent system.

Example 4 : Does the system of simultaneous equations 3x − 2y − 5 = 0 and 3x − 2y + 5 = 0 have a solution?

Solution : Let us try and solve this pair of equations.

Now 3x − 2y − 5 = 0 ⇒ x = (2y + 5)/3.

Substituting this value of x in the equation 3x − 2y + 5 = 0, we get

$3\left(\frac{2y+5}{3}\right) - 2y + 5 = 0 \Rightarrow 10 = 0$, which is a contradiction. Thus, the given system of linear equations is inconsistent.

Try the following exercise now.

E 5) Are the following pairs of simultaneous linear equations consistent? If so, obtain a solution.

a) $x - y = 6$ and $2x = 12 + 2y$,

b) $x + y - 3 = 0$ and $2x + 2y + 5 = 0$.

Let us now look at polynomials of degree two.

3.2.3 Quadratic Equations

A polynomial of degree two, like $2x^2 - 5$ or $xy + y^2 - x$, is called a quadratic polynomial.

If we set a quadratic polynomial equal to zero, we get a quadratic equation. Thus, a **quadratic equation** in one variable, x, is an equation of the form

$ax^2 + bx + c = 0$,

with $a \neq 0$, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$. (If $a = 0$, the equation reduces to a linear equation.) This equation is also called the **standard form of the quadratic equation**, since all the non-zero terms are on the left hand side.

α is called a **root** of the quadratic equation $ax^2 + bx + c = 0$ if $a\alpha^2 + b\alpha + c = 0$. For instance, let us consider the equation

$x^2 - x - 12 = 0$.

If we put $x = 4$ in it, then we get $(4)^2 - 4 - 12 = 0$. So $x = 4$ is a root of the given quadratic equation.

You may often need to determine the roots of a quadratic equation, as in the case of genetic problems. You know, from Section 3.2.1, that the quadratic equation $ax^2 + bx + c = 0$ can have 2 roots at the most. These roots are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

How do we get this expression? We get it by "completing the square", a method known to the ancient Babylonians also. In this method we rewrite the given equation as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Leftrightarrow x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Leftrightarrow x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \left(\text{adding } \frac{b^2}{4a^2} \text{ to both sides}\right)$$

$$\Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 \quad (\text{since } m^2 + 2mn + n^2 = (m+n)^2)$$

Taking square roots on both sides, we obtain

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, the two roots of the given equations are $x = x_1$ and $x = x_2$, where

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The real number $b^2 - 4ac$ is called the **discriminant** of the given equation. We denote it by D.

If $D < 0$, then $\sqrt{D}$ is not a real number. Therefore, the given equation has no real roots.

If $D = 0$, the given equation has **two real roots,** both being equal to $-b/2a$.

If $D > 0$, the given equation has **two distinct real roots,** which are x_1 and x_2 as given above.

Let us take some examples to illustrate these situations.

Example 5 : Solve the equation

$3x^2 + 2x - 2 = 0$

Solution : The given equation is in the standard form. Putting a = 3, b = 2, c = – 2 in the quadratic formula, we get

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4 \times 3(-2)}}{2 \times 3} = \frac{-2 \pm \sqrt{4+24}}{6} = \frac{-2 \pm \sqrt{28}}{6}$$

$$= \frac{-1 \pm \sqrt{7}}{3}$$

Thus, the given equation has two distinct roots, namely,

$$x_1 = \frac{-1 + \sqrt{7}}{3}, x_2 = \frac{-1 - \sqrt{7}}{3}$$

(Notice that D = 28 > 0 in this case.)
Both roots can be checked by substituting them in the given equation.

Example 6: Consider the following genetic problem of recombination of chromosomes: Let p and 1– p be the recombination and non-recombination fractions, respectively, where we assume $0 \le p \le 1/2$. How does p (1– p) behave?

Solution : Let c = p (1 – p). Since $0 \le p \le 1/2$, $1 - p > 0$. ∴ $c \ge 0$. We want to know if, for a given value of c, there exist corresponding values of p and, if so, what they are. So we look at the quadratic equation c = p (1– p), that is,

$p^2 - p + c = 0$.

Using the quadratic formula, the roots of this equation are

$$p = \frac{1 \pm \sqrt{1-4c}}{2}$$

If 1 – 4c > 0, then p could be $\frac{1 + \sqrt{1-4c}}{2}$ or $\frac{1 - \sqrt{1-4c}}{2}$. But, since $p \le 1/2$, we can't admit the first solution. Thus, the only solution is $p = \frac{1 - \sqrt{1-4c}}{2}$

If 1 – 4c = 0, that is, c = 1/4, then we get the solution to be p = 1/2.

If 1 – 4c < 0, then the given equation has no solution.

While solving quadratic equations we need not always use the quadratic formula. In some equations in the following exercises, you could use simpler ways of obtaining the roots.

E6) Solve the following equations:

a) $2x^2 = 3$, b) $-5x + 6x^2 = 2x$, c) $2x + 6/x = 5$, d) $(2 - x)(3 + x) = 4x^2$.

E7) The sum of a positive integer and its square is 30. Find the integer.

(**Hint** : Let the integer be equal to x.)

Now, what happens if we know that the two roots of a quadratic equation are p and q but we don't know the equation? We can find the equation very easily. It will be equivalent to the equation (x – p) (x – q) = 0, since its only factors are x – p and x – q. Thus, the required equation is (x – p) (x – q) = 0, that is, $x^2 - (p + q)x + pq = 0$.

Thus, p and q are roots of $ax^2 + bx + c = 0$, a, b, c ∈ **R** if and only if p + q = – b/a and pq = c/a.

Thus, the

sum of the roots $= -\dfrac{\textbf{coefficient of x}}{\textbf{coefficient of } \mathbf{x^2}}$

product of the roots $= \dfrac{\textbf{constant term}}{\textbf{coefficient of } \mathbf{x^2}}$

This will help you in solving the following exercise.

E 8) Are 2 and 3 roots of the equation $2x^2 - 10x = 12$?

Now let us see how we can graphically solve a quadratic equation. In Unit 2 you came across the graph of the power function $y=x^2$. It was a parabola. The **graph of any quadratic function $y=ax^2 + bx + c$ is a parabola.** The roots of the equation are the points of intersection of the parabola and the x-axis.

When $a > 0$, the graph of $y = ax^2 + bx + c$ is a parabola which opens upwards. The graphs will be similar to either of the ones given in Fig. 2, depending on the value of D, the discriminant. **The lowest points occur at the value of $x = -b/2a$.**

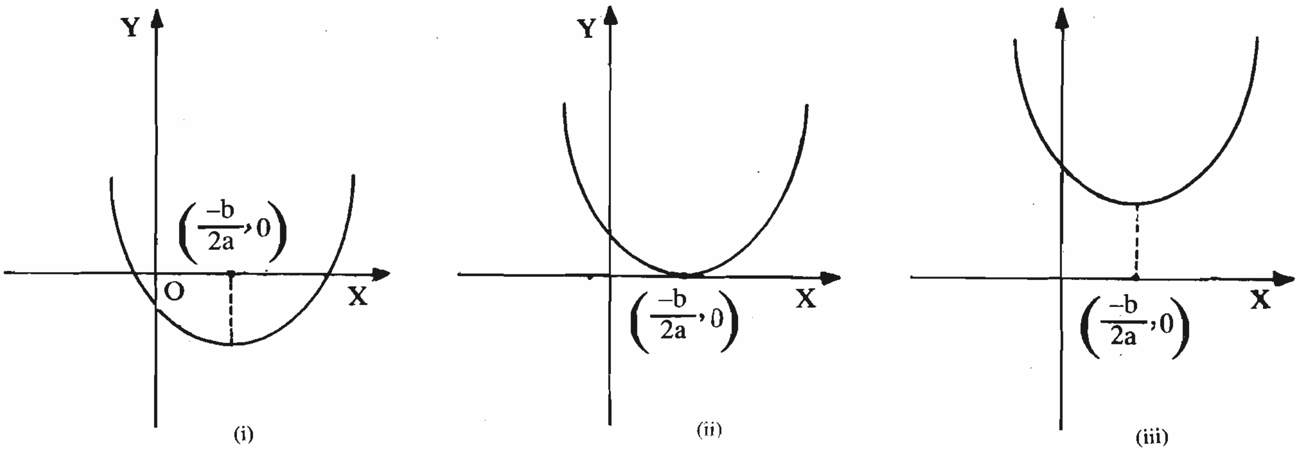

Fig. 2 : The graph of $ax^2 + bx + c = 0$ when

(i) $D > 0$, (ii) $D = 0$, (iii) $D < 0$

In Fig. 2 we have assumed that $-b/2a$ is a positive number. But it can also be negative. In this case the graphs will shift partially or wholly to the left of the y-axis, as in the following example.

Example 7: Draw a rough graph of $y = x^2 + 2x + 1/4$.

Solution : Now, for $x^2 + 2x + 1/4 = 0$, $a = 1$, $b = 2$, $c = 1/4$.

Therefore, the roots of this equation are

$$x = \frac{-2 \pm \sqrt{4-1}}{2} = \frac{-2 \pm \sqrt{3}}{2} = -1 \pm \frac{\sqrt{3}}{2} = -.134, -1.866.$$

The lowest point of the graph we want will be at the point $(-1, -3/4)$. So, we can give a rough sketch (Fig. 3).

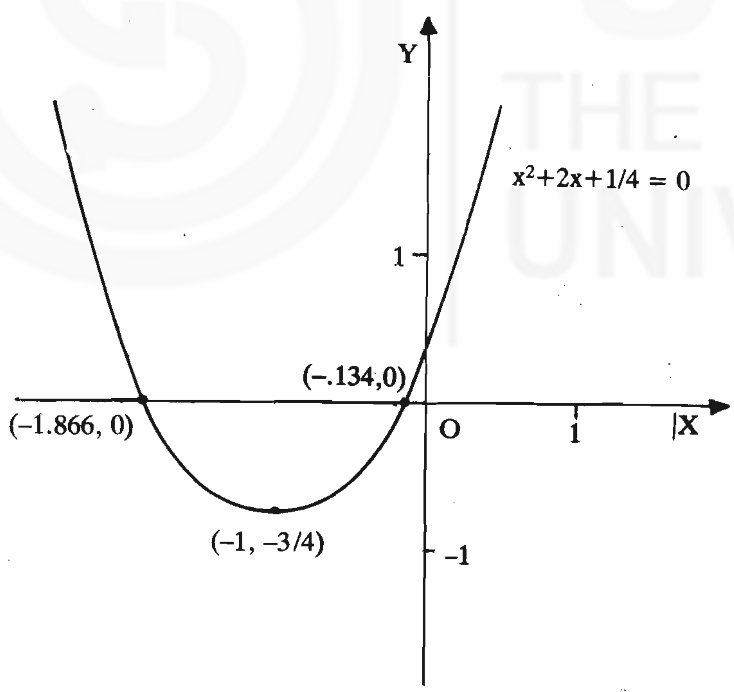

Fig. 3 : $x^2 + 2x + 1/4 = 0$

Similarly, if $a < 0$ then the parabolas open downwards, with the highest points being at $x = -b/2a$.

Try the following exercise now.

E 9) Draw the graph of the polynomial.

a) $x^2 + x - 2$,

b) $-(x - 5)^2$.

We will now look at another part of basic algebra.

3.3 SEQUENCES AND SERIES

In this section we will first discuss sequences, which are sets of numbers arranged in some order. Then you will study about series, which are the sums of numbers in a sequence.

3.3.1 Sequences

Consider the growth of a child. Let its weight be w_1 at the age of one year, w_2 at the age of two years, and so on. Thus, we get a set of numbers $w_1, w_2, w_3, \ldots$, which are arranged in a definite order. We can also present this information as a function $f : \mathbf{N} \to \mathbf{R} : f(n) = w_n$. Then $f(1) = w_1$, $f(2) = w_2$, etc. This function is an example of a sequence. We give the following definition.

Definition: Consider a function $f : D \to \mathbf{R}$ where $D \subseteq \mathbf{N}$. The arrangement of the elements of its range, in some order, is called a **sequence**. Each number in a sequence is called a **term** of the sequence.

If D is infinite, f will define an **infinite sequence**. If D is finite, f will define a **finite sequence**.

In the example above we have an infinite sequence, since the domain of f is **N**, which is infinite. Its first term is w_1, second term w_2, and so on.

Now look at the sequence 2, 4, 6, 8, 10 given by the function $f : \{1, 2, 3, 4, 5\} \to \mathbf{R} : f(x) = 2x$. Here, the domain of f is finite, and we get a finite sequence, consisting of only 5 terms.

Sometimes the terms in a sequence may have no apparent order at all. For example, the sequence 1, 5, 13, 100, 150, ... doesn't seem to have an order, but it is the range of a function f for which $f(1) = 1$, $f(2) = 5$, etc.

We can also have finite or infinite sequences in which all the terms are the same, like 1, 1, 1, ...; or sequences in which the terms may only take a finite number of values, like 1, 2, 0, 1, 2, 0, 1, 2, 0, ...

There are various ways of briefly denoting a sequence. We can denote the sequence $a_1, a_2, \ldots$ by $< a_n | n \in \mathbf{N} >$ or $< a_n >_n$. Here a_n is the **n^{th} term**, or the **general term**, of the sequence.

If the sequence is finite, say $a_1, a_2, \ldots, a_s$, then we denote it by $<a_n>_{n=1}^{s}$. If the sequence is infinite, then we denote it by $< a_n >_{n=1}^{\infty}$, or simply $< a_n >$.

Now, if we know the formula for the nth term of a sequence, then we can list out the whole sequence. For example, the sequence $< a_n >_n$, for which $a_n = 3n - 1$, is 2, 5, 8, ...

A very well-known sequence is the Fibonacci sequence 1, 1, 2, 3, 5, 8, ..., where each term is the sum of the two preceding terms. Thus, the rule is $a_n = a_{n-1} + a_{n-2}$.

The Fibonacci sequence is named after the Italian mathematician Fibonacci (1170– 1230).

Now try the following exercises.

E 10) Write the first four terms of the sequence with general term a_n, where

a) $a_n = (-1)^n$,

b) $a_n = 3 + (n - 1)5$,

c) $a_n = 2^{n-1}$.

E 11) Given $a_n = (-1)^n + (-1)^{n+1}$, write $< a_n >_{n=1}^{\infty}$.

Let us now look at two particular types of sequences.

3.3.2 AP and GP

In this sub-section we will discuss two types of sequences that are defined according to the rules by which their terms progress. We first discuss arithmetic progressions (AP) and then geometric progressions (GP).

Arithmetic Progression: In E 10) (b) you see that the form of the sequence is a, a + d, a + 2d, ... (where a = 3 and d = 5). Such a sequence, in which the difference $a_{n+1} - a_n$ is a constant for all values of n, is called an **arithmetic progression** (or **arithmetic sequence), AP** for short. The difference $a_{n+1} - a_n$ is called the **common difference.** For example, the sequence 3, 5, 7, 9, ... has common difference 2, and the sequence 5, 3, 1, – 1, ... has common difference – 2.

The nth term of an AP whose first term is a and common difference is d is

$a_n = a + (n-1)d$, n = 1, 2, ...

Remark: In an AP each term a_n is the arithmetic mean of the terms a_{n-1} and a_{n+1} $\left(\text{since } a_n = \frac{a_{n+1} + a_{n-1}}{2}\right)$.

Note that an arithmetic progression need not be infinite. It may be a finite sequence, like 1, 3, 5, 7, 9, where the first term a = 1 and common difference d = 2.

Also, a constant sequence a, a, a, ... can be regarded as an arithmetic progression with common difference d = 0.

Let us look at an example that involves an AP.

Example 8: It is said that the mathematician De Moivre died of an AP! He suddenly began to need 15 minutes more sleep each night. When the terms of this AP reached 24 hours, he died! How many days did his illness last?

Solution : Now 24 hours is 24 × 60 (= 1440) minutes. The problem is to find which term of the AP 15, 30, 45, ... is 1440. That is, we want to find n for which 15 + (n – 1) 15 = 1440. On solving this equation, we get n = 96.

Try the following exercises now.

E 12) Find the formula for the nth term a_n in the following arithmetic progressions.

a) –2, 3, 8, 13, ...

b) –2, –4, – 6, –8, ...

E 13) The third and seventh terms of an arithmetic progression are 10 and 34, respectively. Find its first term and the common difference. Determine the 5th term also.

(**Hint** : Let the AP be $< a_n >$. Then $a_n = a + (n-1)d$, where $a = a_1$.)

Now let us consider geometric progressions.

Geometric Progression: Consider the sequence given in E 10) (c). It is 1, 2, 2^2, 2^3, ... Each term is obtained from the previous one by multiplying it by a constant number, 2. This is an example of a geometric progression.

Definition : A geometric progression (or a **geometric sequence**) is a sequence $< a_n >$ in which $\frac{a_{n+1}}{a_n}$ is constant, $\forall$ n ∈ N. This constant is called the **common ratio**, and is usually denoted by r. Of course, r 0.

A geometric progression is briefly denoted by **GP**.

The GP, whose first term is a and common ratio is r, is a, ar, ar^2, ar^3, ...

The nth term of this GP is ar^{n-1}.

Remark : Each term a_r of a GP is the geometric mean of the terms a_{n+1} and a_{n-1} (since

$a_n = \pm\sqrt{a_{n+1}a_{n-1}}$).

A GP need not be infinite. The sequence 2, 6, 18, 54 is also a GP with common ratio 3.

Also, a constant sequence

a, a, a, ..., where $a \neq 0$,

can be regarded as a geometric progression with r = 1.

GPs are used while studying cell division or radioactive decay. Consider the following example.

Example 9: A cell divides into two cells in a time interval t. After how long will it have subdivided into 256 cells?

Solution : Here we have a GP 1, 2, 2^2, ... whose nth term is 2^{n-1}. Now, $2^{n-1} = 256$ gives us n = 9. Thus, after the time interval 9t, the cell will have subdivided into 256 cells.

Try the following exercise now.

E 14) The first and fourth terms of a geometric progression are 8 and 27, respectively. Determine the common ratio r.

Let us now look at the sum of the terms of a sequence.

3.3.3 Series

A series

$a_1 + a_2 + a_3 + ...$

is an expression which represents the sum of all the terms of a sequence $< a_n >_n$.

For instance, the series 1 + 4 + 7 + 10 + ... corresponds to the sequence 1, 4, 7, 10, ...

We denote the series corresponding to $< a_n >_n$ by $\sum_n a_n$. (Σ is the Greek letter 'sigma' and stands for sum.) If $< a_n >_n$ is a finite sequence consisting of s terms, then the corresponding series is $a_1 + a_2 + \cdots + a_s = \sum_{n=1}^{s} a_n$. If $< a_n >$ is infinite, then the series $a_1 + a_2 + ...$ is called an **infinite series**, and can be briefly written as $\sum_{n=1}^{\infty} a_n$.

Remark : A finite series will always have a sum. But an infinite series may not have a sum. Over here we will not go into the conditions under which the sum of an infinite series exists.

Let us see what the sum of an AP is.

Definition : Given an AP, a, a + d, a + 2d, ..., there corresponds a series a + (a + d) + (a + 2d) + ... This series is known as an **arithmetic series**.

Let us find S_n, the sum of the first n terms of this arithmetic series. Now,

$S_n = a + (a + d) + (a + 2d) + ... + [a + (n - 1) d]$ .... (1)

Writing the right hand side in reverse order we get

$S_n = [a + (n - 1) d] + [a + (n - 2) d] + ... + a$ .... (2)

Adding (1) and (2), we get

$2S_n = [2a + (n - 1)d] + [2a + 9n - 1)d] + ... + [2a + (n - 1)d]$ (n times)

$= n[2a + (n - 1)d]$

Thus, $S_n = \frac{n}{2}[2a + (n - 1)d]$, that is,

This formula for S_n was given by Leonardo of Pisa in 1202.

the sum of n terms of an AP with first term a and common difference d is

$S_n = \frac{n}{2}[2a + (n - 1)d]$.

This formula may also be rewritten as

$S_n = \frac{n}{2}[a + \{a + (n - 1)d\}] = \frac{n}{2}(a_1 + a_n)$

Let us look at some examples involving arithmetic series.

Example 10 : Find the sum of the series 1 + 6 + 11 + ... upto 5 terms.

Solution : The given series is an arithmetic series for which a = 1 and d = 5. Thus, the required sum is

$$S_5 = \frac{5}{2}[2 + 4 \times 5] = 55.$$

Example 11: The 6th and 10th terms of an AP are 22 and 34, respectively. Find the sum of its first 16 terms.

Solution : Suppose the AP is $< a_n >$, where

$a_n = a + (n - 1)d$. Then $a_6 = a + 5d = 22$ and

$a_{10} = a + 9d = 34$. Solving these two equations for a and d, we get a = 7 and d = 3.

Therefore , the sum of the first 16 terms of the AP is

$$S_{16} = \frac{16}{2}[2 \times 7 + 15 \times 3] = 472.$$

Try the following exercises now.

E 15) Find the sum of the following series.

a) $(-3) + (-1) + 1 + 3 + ...$ to 20 terms.

b) $1 + 2 + 3 + 4 + ...$ to n terms.

E 16) The sum of an arithmetic series with 4 and 76 as its first and last terms is 1920. Determine the number of terms in the arithmetic series.

E 17) Show that the sum of the first n terms of the arithmetic series $\log 3 + \log 6 + \log 12 + ...$ is

$$n \log 3 + \frac{n(n-1)}{2} \log 2.$$

Now let us consider the series whose terms form a GP.

Definition : A series corresponding to a GP is a **geometric series.** This series is a **finite series** or an **infinite series** according to whether the corresponding GP is finite or infinite.

Let us find the sum of the first n terms of the geometric sequence a, ar, ar^2, ..., when $r \neq 1$. Let

$S_n = a + ar + ar^2 + ... + ar^{n-1}$

Multiplying both sides by r, we obtain

$r S_n = ar + ar^2 + ... + ar^{n-1} + ar^n$.

Hence,

$S_n - rS_n = a - ar^n$, which gives us

$$S_n = \frac{a(1-r^n)}{1-r}, r \neq 1.$$

If r = 1, then $S_n = a + a + a + ... + a$ (n times) = na.

Thus, **the sum up to n terms of the GP whose first term is a and common ratio is r is**

$$S_n = \begin{cases} a_n \dfrac{(1-r^n)}{1-r}, & \text{if } r \neq 1. \\ na, & \text{if } r = 1. \end{cases}$$

Consider the following example.

Example 12: Given the first term a = 4 and common ratio r = 1/2, of a geometric progression,

i) write the first 5 terms of the progression.

ii) find the sum of its first 10 terms.

Solution: i) The first five terms of the geometric progression are 4, 2, 1, 1/2, 1/4.

ii) Here $r \neq 1$.

$$\therefore S_{10} = \frac{4[1-(1/2)^{10}]}{1-(1/2)} = 8[1-(1/2)^{10}] = \frac{1023}{128}$$

Try this exercise now.

E 18) Show that $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n-2}} + \frac{1}{2 \times 3^{n-2}}$

(This problem occurs in Ganit Sara Sangraha written by Mahaviracharya in 850 A.D.)

In an earlier remark we said that an infinite series may not have a sum. In the case of a geometric series, the sum will exist if $|r|<1$. We state the following result, the proof of which is beyond the scope of this course.

The first known calculation of the sum of an infinite geometric series was done by the ancient Greek mathematician Archimedes (287– 212 B.C.).

If $|r| < 1$, then the sum of the geometric series

$\mathbf{a + ar + ar^2 + \cdots}$ **is** $\mathbf{\frac{a}{1-r}}$.

Consider the following examples.

Example 13: Obtain the sum of the series in Example 12.

Solution : Using the formula, the sum $= \frac{4}{1 - \frac{1}{2}} = 8$.

Example 14: Determine the sum of the series

$3 + \sqrt{3} + 1 + \ldots$ to n terms, and to infinity.

Solution : The common ratio for the given series is $1/\sqrt{3}$.

Thus, the sum up to n terms is $\frac{3[1 - (1/\sqrt{3})^n]}{1 - 1/\sqrt{3}}$

The sum to infinity exists since $|1/\sqrt{3}| < 1$.

It is $\frac{3}{1 - 1/\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{3} - 1}$.

Example 15: Express the recurring decimal $.\dot{3} = .333\ldots$ as an infinite geometric series, and hence reduce it to a rational fraction.

Solution : Now $.\dot{3} = .333\ldots\ldots = .3 + .03 + .003 + \ldots$, which is an infinite geometric series having .1 as its common ratio.

Therefore,

$.\dot{3} = \frac{.3}{1 - .1} = \frac{1}{3}$.

Try this exercise now.

E 19) The first term of a geometric series is 100 and the third term is 1. Find the sum to infinity of each of the two possible series.

And now let us discuss some ways of counting.

3.4 PERMUTATIONS AND COMBINATIONS

Consider the following situation: Two men and three women are standing for elections for the posts of president and secretary of a society. Each person can hold one post only. Moreover, the president must be a woman. In how many ways can the posts be filled?

This section discusses the number of all possible arrangements and choices in such cases. We start with the principle of counting.

3.4.1 Principle of Counting

Suppose a person wants to travel from Rampur to Jaipur by road. The road map shows that he must go via Dinapur. There are three routes for going from Rampur to Dinapur and two routes for going from Dinapur to Jaipur (see Fig. 4).

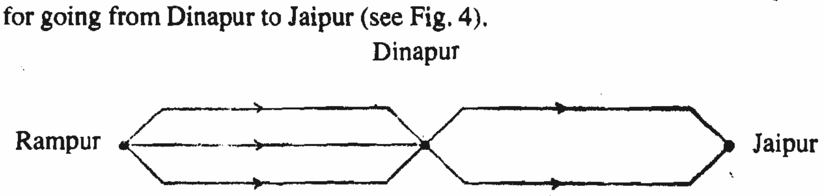

Fig. 4: 6 different ways of getting from Rampur to Jaipur

Obviously, he can go from Rampur to Dinapur in three ways, as he can choose any one of the three available routes. Also, he can go from Dinapur to Jaipur in two different ways, as he can take any one of the two available routes. Thus, he can go from Rampur to Jaipur in 6 possible ways.

In such situations the principle of counting tells us the number of possibilities without actually enumerating them. Let us see what the principle is.

The Fundamental Principle of Counting: If an event can occur in m different ways, and, after it has happened, another event can occur in n different ways, then the total number of different ways in which both events can occur is mn.

In the above example there are 3 routes from Rampur to Dinapur, and 2 routes from Dinapur to Jaipur. Thus, the total number of routes from Rampur to Jaipur is $3 \times 2 = 6$.

Let us look at another example.

Example 16: How many 3 digit numbers can be formed if the digits can be repeated?

Solution : The leftmost digit has only 9 choices — 1, 2, ..., 9. The middle digit and the rightmost digit have 10 choices each — 0, 1, 2, ..., 9.

Thus, the total number of 3 digit numbers is $9 \times 10 \times 10 = 900$.

Try the following exercise now.

E 20) How many 4 digit numbers can be formed if the digits cannot be repeated?

We will now introduce you to the factorial notation.

3.4.2 Factorial Notation

A useful notation that we shall be using in the following sub-sections is that of the factorial. We denote the **product of the first n positive integers by n!**, or $\underline{|n}$, to be read as **n factorial**. Thus,

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6.$

In general,

$n! = n(n-1)(n-2) \ldots . 3.2.1$

Note that $n! = n.(n-1)!$

$= n(n-1)(n-2)!$

$= n(n-1)(n-2) \ldots\ldots (n-r+1)(n-r)!$

By convention $0! = 1$.

The following exercises will help you in getting used to the factorial notation.

E 21) Evaluate

a) $5!$, b) $\frac{8!}{6!}$, c) $3! + 2!$.

E 22) Obtain $\frac{n!}{(n-r)!}$ and $\frac{n!}{r!(n-r)!}$ for

a) $n = 5, r = 2$,

b) $n = 5, r = 3$.

Note that E 21) (a) and E 21 (c) show us that $3! + 2! \neq 5!$.

We will use the factorial notation for obtaining the number of permutations and combinations possible in a given situation.

$(n + m)! \neq n! + m!$, in general.

Let us first see what a permutation is.

3.4.3 Permutations

Suppose 5 people want to spend the night in a hotel and only three rooms are free for them. In how many different ways can the people be allotted the rooms? The first room can be taken by any of the 5 people, the second room by any of the remaining 4, and the last room by any of the remaining 3 people. Thus, the total number of possibilities is $5 \times 4 \times 3 = 60$. Two people remain without a room.

In general, suppose we are given r distinct spaces and n different objects (n > r), to put into these spaces, one object per space. Then we can fill the first space in n ways, the second space in (n – 1) ways, ..., and the r th space in (n – r + 1) ways. So, the total number of ways in which we can arrange n objects, r at a time, is n (n – 1) (n – 2) .... (n – r + 1). Each such arrangement of n objects taken r at a time is a **permutation.** We denote the total number of such permutations by **P (n, r)** (or $^n\mathbf{P}_r$).

Thus,

$$P(n,r) = n(n-1)\cdots(n-r+1) = \frac{n(n-1)\ldots(n-r+1)(n-r)(n-r-1)\ldots 3.2.1}{(n-r)(n-r-1)\ldots 3.2.1}$$

This tells us that $\mathbf{P(n, r)} = \frac{n!}{(n-r)!}$.

If r = 0, we get $P(n, 0) = \frac{n!}{n!} = 1$.

Also, $P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$.

See if you can do the following exercise.

E 23) Consider a geographic map of 3 countries. Each country is to be painted with a different colour. There are 6 colours available. In how many ways can the painting be done?

Another problem of counting deals with selections, where the order of arrangement is not relevant.

3.4.4 Combinations

At the beginning of Section 3.4.3 we gave a situation where only 3 rooms were available for 5 people. If we call the three people who got the rooms, a, b and c, then we considered all the following cases as distinct: abc, acb, bac, bca, cab, cba.

Thus, whatever way we chose the 3 people, our selection was counted 6 times, depending on the order in which they were given their rooms. Now suppose we aren't interested in the order in which the people are given the rooms. Then we must divide the total number of ways of allotting the rooms by 6. So there are 60/6 = 10 ways in which 5 people can be allotted 3 rooms, if the order of allotment is not taken into consideration. We call each such possibility, in which the order is disregarded; a combination.

Definition : An arrangement of n objects, taken r at a time, in which the order of arrangement is not considered, is called a **combination** of n objects, taken r at a time. The total number of all such combinations is denoted by **C(n, r)** (or $^n\mathrm{C}_r$, or $\binom{n}{r}$).

Now each combination of n objects, taken r at a time gives us r! distinct permutations of these objects, taken r at a time.

So,

$r!\,C(n, r) = P(n, r)$

$$\Rightarrow \mathbf{C(n,r)} = \frac{\mathbf{1}}{\mathbf{r!}}\mathbf{P(n,r)} = \frac{\mathbf{n!}}{\mathbf{r!(n-r)!}}.$$

In particular,

$$C(n,0) = \frac{n!}{0!\,n!} = 1, \text{ and}$$

$$C(n,n) = \frac{n!}{n!\,0!} = 1$$

Let us look at an example involving combinations.

Example 17: A student is asked to choose any 3 courses out of 12 courses being offered by the School of Sciences. In how many ways can she do this?

Solution : The answer is C (12, 3), since the order in which the courses are chosen is not important. Now,

$$C(12, 3) = \frac{12!}{9!\,3!} = \frac{12 \times 11 \times 10 \times 9!}{9!\,3!} = \frac{12 \times 11 \times 10}{3 \times 2} = 220.$$

Try the following exercise now.

E 24) In how many ways can a person choose 5 questions to answer out of 8 questions in an exam?

There are several formulas concerning C (n, r) which may be useful when we have to evaluate C (n, r). Some are obvious, while the others require proof. We give 5 formulas here, and prove the first two. The proof of the third is a little involved, so we shall not do it here. We ask you to prove the last two formulas (see E 25).

i) $C(n, r) = \frac{n}{r} C(n-1, r-1).$

Proof: $C(n, r) = \frac{n!}{r!(n-r)!}$

$$= \frac{n(n-1)!}{r(r-1)!\,(n-1-r+1)!} = \frac{n}{r} C(n-1, r-1).$$

ii) $C(n, r) = C(n-1, r-1) + C(n-1, r)$

Proof : $C(n-1, r-1) + C(n-1, r) = \frac{(n-1)!}{(n-r)!\,(r-1)!} + \frac{(n-1)!}{(n-r-1)!\,r!}$

$$= \frac{(n-1)!}{(n-r)!\,r!}(r+n-r)$$

$$= \frac{n!}{r!\,(n-r)!} = C(n, r)$$

iii) $C(n, 0) + C(n, 1) + \ldots + C(n, n) = 2^n$

iv) $C(n, r) = C(n, n-r)$

v) $C(n, r) = \frac{n-r+1}{r} C(n, r-1)$

E 25) Prove the formulas (iv) and (v), given above.

Now suppose we have to arrange n objects, r at a time, and m other objects, s at a time. How many such combinations are possible? The total number will be $C(n, r) \times C(m, s)$.

Consider the following example.

Example 18: For an experiment I need to choose 4 animals from 3 monkeys and 4 guinea pigs. In how many ways can I do this so as to have 2 monkeys and 2 guinea pigs?

Solution : Two monkeys can be chosen in C(3, 2) ways. Two guinea pigs can be chosen in C(4, 2) ways. Thus, the total number of choices is $C(3, 2) \times C(4,2) = 3 \times 6 = 18$

Try the following exercise now.

E 26) Out of 10 male and 7 female cats, a sample consisting of 2 males and 2 females was chosen for an experiment. In how many ways could this be done?

And now we will discuss a result that you may have heard of very often.

3.5 BINOMIAL THEOREM

A binomial is a polynomial with two terms, like x + y. (Here x and y denote the terms of the binomial. Don't confuse them with the variables x and y that you have been using earlier.)

We see that

$(x + y)^1 = x + y = C(1, 0)\,x + C(1, 1)\,y$

$(x + y)^2 = x^2 + 2xy + y^2 = C(2, 0)\,x^2 + C(2, 1)\,xy + C(2, 2)\,y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$= C(3, 0)\, x^3 + C(3, 1)\, x^2y + C(3, 2)\, xy^2 + C(3, 3)\, y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$= C(4, 0)\, x^4 + C(4, 1)\, x^3y + C(4, 2)\, x^2y^2 + C(4, 3)\, xy^3 + C(4, 4)\, y^4$

You can see that we have expressed the numerical coefficients on the right hand side in terms of C(n, r). A natural question is: Does a similar formula hold for $(x + y)^n$, where n is any positive integer? The answer is given by the binomial theorem.

Theorem 1 (Binomial theorem): For any positive integer n,

$(x + y)^n = C(n, 0)x^n + C(n, 1)\, x^{n-1}y + C(n, 2)\, x^{n-2}\, y^2 + \ldots + C(n, r)x^{n-r}\, y^r + \ldots + C(n, n)\, y^n.$

$$= \sum_{r=0}^{n} C(n, r)\, x^{n-r} y^r$$

Since the proof of this theorem is rather technical, we shall not give it over here.

The expansion of $(x + y)^n$ given in Theorem 1 is called its **binomial expansion**.

We now make some observations about any binomial expansion. While going through these remarks you can keep the expansions of $(x + y)$, $(x + y)^2$, $(x + y)^3$ and $(x + y)^4$ in mind.

Remarks : i) The expansion of $(x + y)^n$ contains n + 1 terms.

ii) The sum of the exponents of x and y in each term of the expansion $(x + y)^n$ is n.

iii) The numerical coefficients of the terms containing $x^r\, y^{n-r}$ and $x^{n-r}\, y^r$ are equal (since $C(n, r) = C(n, n - r)$).

iv) The numerical coefficients of the terms in the expansion of $(x + y)^n$ form the nth row of the triangle given below.

1

1 1

1 2 1

1 3 3 1

. . . .

1 C(n, 1) C(n, 2) . . . . . . C(n, n-1) 1

This triangle is called **Pascal's triangle**, after the French mathematician Blaise Pascal (1623 — 1662). In this triangle, all the coefficients on its boundary are 1. Any other coefficient is the sum of the two nearest coefficients that are in the row above it. For example, the second element in the 4th row, is 1 + 2.

Some particular cases of the binomial theorem are of importance. Let us choose 1 + y as our binomial. Then $(1 + y)^n = C(n, 0) + C(n, 1)y + C(n, 2)y^2 + \ldots + C(n, r)\, y^r + \ldots + C(n, n)\, y^n$

Now let us consider powers of the binomial $1 - a$, $a > 0$.

$(1 - a)^n = C(n, 0) - C(n, 1)\, a + C(n, 2)\, a^2 - \ldots + (-1)^n C(n, n)\, a^n$, the last term being positive or negative according as n is even or odd.

Use the binomial theorem to solve the following exercise.

E 27) Expand $(2x + 5)^7$ and $(1/2 + 1/3)^5$.

So far we have discussed the expansion of $(a + b)^n$ for any $n \in \mathbf{N}$. In fact, **the binomial theorem holds true for any $n \in \mathbf{Q}$.**

We end by summarising what we have covered in this unit.

3.6 SUMMARY

In this unit we have covered the following points.

1) Polynomials and operations on polynomials.
2) How to solve a linear equation as well as a system of simultaneous linear equations.
3) How to solve quadratic equations.
4) Definition and examples of a sequence, arithmetic progression and geometric progression.
5) Definition and examples of a series, arithmetic series and geometric series.
6) The principle of counting, permutations and combinations.
7) The binomial theorem.

3.7 SOLUTIONS/ANSWERS

E 1) Except for x/y, all the expressions are polynomials.

E 2) Degree of 0 = 0.
Degree of (9/5) C + 32 = maximum of degree of 9/5 C and degree of 32.
= max. (1, 0) = 1.
Degree of $x^4 + 4x^3 - 3x^2 - x^4 + 1$ = degree of $4x^3 - 3x^2 + 1 = 3$.
Degree of $(x^2 - y) - (x^2 + y)$ = degree of $(-2y) = 1$.

E 3) $x^2 - 9$ can have 2 roots, at the most. Since $(2)^2 - 9 \neq 0$, 2 is not a root of $x^2 - 9$.
Therefore, $(x - 2)$ does not divide $(x^2 - 9)$

E 4) a) Only one solution, $x = 3/2$.
b) Infinitely many solutions, of which one is (0, 2).
c) Only one solution, $y = 7/2$.

E 5) a) $x - y = 6 \Rightarrow x = y + 6$. Substituting this in $2x = 12 + 2y$, we get $2(y + 6) = 12 + 2y \Rightarrow 0 = 0$. This shows that both the equations are equivalent. That is, they represent the same line $x - y = 6$. Hence, every point on this line is a solution for both of them. Thus, a solution could be (0, –6).
b) $x + y - 3 = 0 \Rightarrow x = 3 - y$. Substituting this in $2x + 2y + 5 = 0$, we get $2(3 - y) + 2y + 5 = 0 \Rightarrow 11 = 0$, a contradiction.
Therefore, the given system of equations is inconsistent.

E 6) a) $2x^2 = 3 \Rightarrow x^2 = 3/2 \Rightarrow x = \pm\sqrt{\frac{3}{2}}$.
b) $-5x + 6x^2 = 2x \Rightarrow 6x^2 - 7x = 0 \Rightarrow x(6x - 7) = 0$.
$\Rightarrow x = 0$ or $6x - 7 = 0 \Rightarrow x = 0, 7/6$.
c) $2x + 6/x = 5 \Rightarrow 2x^2 - 5x + 6 = 0$. Hence, by the quadratic formula, we get
$x = \frac{5 \pm \sqrt{25 - 48}}{4}$, neither of which are real.
Therefore, this equation has no real roots.
d) $(2 - x)(3 + x) = 4x^2 \Rightarrow 5x^2 + x - 6 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1 + 120}}{10}$
$\Rightarrow x = 1, -1.2$.

E 7) Let p be the positive integer. Then,
$p^2 + p = 30 \Rightarrow p^2 + p - 30 = 0 \Rightarrow p = \frac{-1 \pm \sqrt{1 + 120}}{2}$
$\Rightarrow p = 5$ or -6.
But p is a positive integer. Therefore, we can't accept the value $p = -6$. Hence, the required integer is 5.

E 8) For the given equation $a = 2, b = -10, c = -12$.
Now, $2 + 3 = -(-10/2)$. But $2 \times 3 \neq c/a$. Hence, 2 and 3 are not roots of the given equation.

E 9) a) The roots of $x^2 + x - 2 = 0$ are $\frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$.

The lowest point of the required parabola is $(-1/2, -9/4)$. Thus, the graph is as shown in Fig. 5.

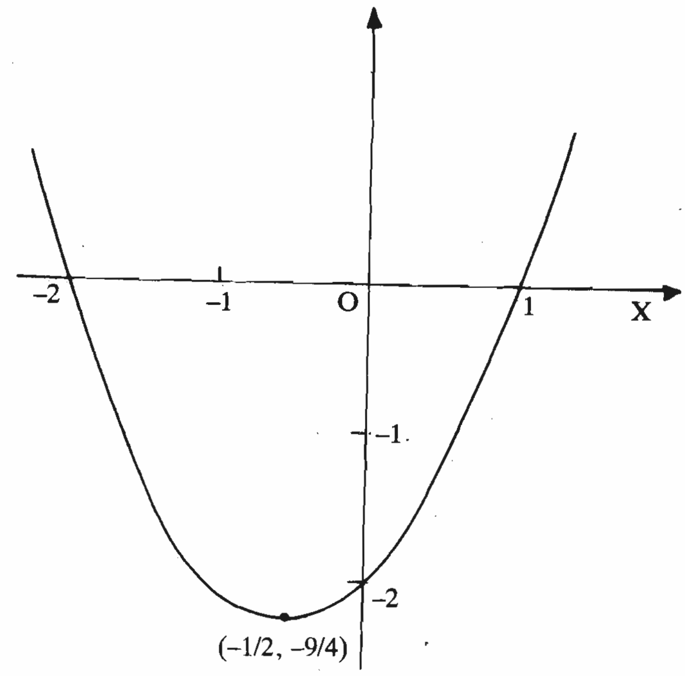

Fig. 5 : $y = x^2 + x - 2$

b) Both the roots of $-(x - 5)^2 = 0$ are 5. Here D = 0 and a <0. Therefore, the graph of $y = -(x - 5)^2 = -x^2 + 10x - 25$ is

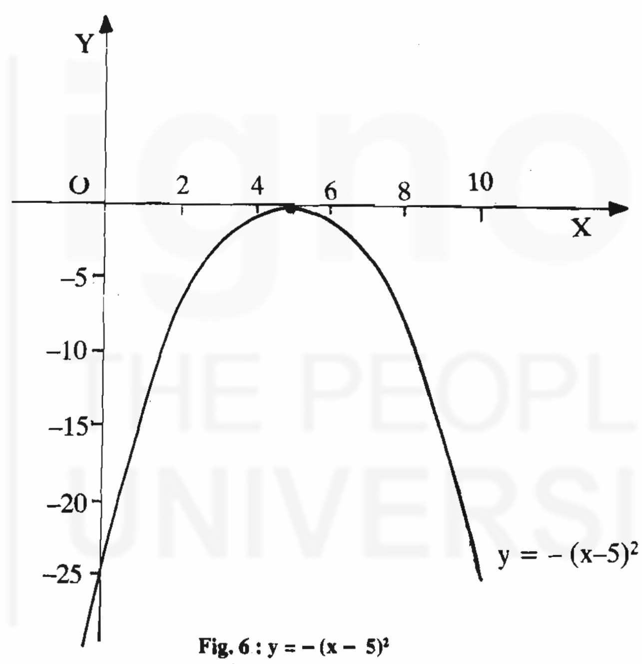

Fig. 6 : $y = -(x - 5)^2$

E 10) a) $-1, 1, -1, 1$; b) 3, 8, 13, 18 ; c) 1, 2, 4, 8.

E 11) $-1 + 1, 1-1, -1 + 1, 1 - 1, \ldots$, that is, 0, 0, 0, ...

E 12) a) $a_n = (-2) + (n - 1) 5$.
b) $a_n = (-2) + (n - 1)(-2)$.

E 13) We know that $a_3 = 10$ and $a_7 = 34$.
$\therefore$ $a + 2d = 10$ and $a + 6d = 34$. Solving these simultaneous equations for a and d, we get
$a = -2, d = 6$.
$\therefore$ $a_5 = a + 4d = 22$.

E 14) Let the GP be $< a_n >_n$. Then $a_1 = 8$ and $a_4 = 27 = a_1 r^3$.
$= 8r^3$. $\therefore$ $r^3 = 27/8 \Rightarrow r = 3/2$.

E 15) a) Here $a = -3$, $d =$ therefore,
$S_{20} = (20/2)\,[(-6) + 19\,(2)] = 320$

b) Here $a = 1$, $d = 1$.
$\therefore\ S_n = (n/2)\,[\,2 + (n-1)] = n\,(n+1)/2$.

E 16) Let s be the number of terms and d be the common difference. Now a = 4.
$\therefore\ 76 = a + (s-1)\,d = 4 + (s-1)\,d \Rightarrow\ (s-1)\,d = 72$.
Also, $1920 = (s/2)\,[\,2a + (s-1)\,d] = (s/2)\,[8 + (s-1)\,d]$.
$= (s/2)\,[8 + 72] = 40s$.
$\therefore\ s = 1920/40 = 48$.

E 17) The common difference is log2.
$\therefore\ S_n = (n/2)\,[2\log 3 + (n-1)\log 2]$.
$= n\log 3 + [n\,(n-1)/2]\log 2$.

E 18) Consider the GP whose first term is 1/3 and common ratio is 1/3. For this,

$$S_{n-2} = \frac{(1/3)[1-(1/3)^{n-2}]}{1-1/3} = \frac{1}{2}[1-(1/3)^{n-2}]$$

That is, $\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-2}} = \frac{1}{2}\left(1 - \frac{1}{3^{n-2}}\right)$

$$\therefore \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-2}}\right) + \frac{1}{2\times 3^{n-2}}$$

$$= \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{3^{n-2}}\right) + \frac{1}{2\times 3^{n-2}} = 1.$$

E 19) Let the series be $\Sigma\, a_n$.
Now $a_1 = 100$ and $a_3 = 1 = a_1 r^2 = 100\,r^2$. $\therefore r = \pm\frac{1}{10}$.
Therefore, the series can be
$100 + 10 + 1 + .1 + \ldots$
or $100 - 10 + 1 - .1 + .01 - .001 + \ldots$
In both cases $|r| = 1/10 < 1$. So, the sum to infinity exists in both cases.

In the first case the sum is $\frac{100}{1-1/10} = \frac{1000}{9}$

In the second case the sum is $\frac{100}{1+1/10} = \frac{1000}{11}$

E 20) The leftmost digit has 9 choices. Once this digit is fixed, the next digit will again have only 9 choices, since we can't repeat a digit. Similarly, the next digit will have 8 choices, and the rightmost digit will have 7 choices.
Thus, the total number of digits will be $9 \times 9 \times 8 \times 7 = 4536$.

E 21) a) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

b) $\frac{8!}{6!} = \frac{8 \times 7 \times 6!}{6!} = 8 \times 7 = 56$.

c) $3! + 2! = 6 + 2 = 8$.

E 22) a) $\frac{5!}{3!} = \frac{5 \times 4 \times 3!}{3!} = 5 \times 4 = 20$.

$\frac{5!}{2!3!} = \frac{1}{2}\left(\frac{5!}{3!}\right) = \frac{1}{2} \times 20 = 10$.

b) $\frac{5!}{2!} = 60$, $\frac{5!}{3!2!} = 10$.

E 23) The first country has 6 choices, the next country has 5 choices and the third country has 4 choices. Thus, the total number of possibilities is $6 \times 5 \times 4 = P\,(6,3)$.

E 24) In $C(8, 5) = 56$ ways.

E 25) iv) $C(n, r) = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)![n-(n-r)]!} = C(n, n-r)$

v) $\frac{n-r+1}{r} C(n, r-1) = \frac{n-r+1}{r} \frac{n!}{(r-1)!\,(n-r+1)!} = \frac{n!}{r!\,(n-r)!}$

$= C\,(n, r)$.

E 26) 2 males can be picked in ${}^{10}C_2$ ways.

2 females can be picked in ${}^{7}C_2$ ways.

Thus, the total number of ways is ${}^{10}C_2 \times {}^{7}C_2$

E 27) $(2x + 5)^7 = (2x)^7 + 7\,(2x)^6.5 + 21\,(2x)^5.5^2 + 35\,(2x)^4.5^3 + 35\,(2x)^3.5^4 + 21\,(2x)^2.5^5 + 7\,(2x).5^6 + 5^7.$

$$\left(\frac{1}{2}+\frac{1}{3}\right)^5 = \left(\frac{1}{2}\right)^5 + 5\left(\frac{1}{2}\right)^4.\left(\frac{1}{3}\right) + 10\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2 + 10\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5.$$