UNIT 2 GRAPHS AND FUNCTIONS

Structure

2.1 INTRODUCTION

In Unit 1 you were introduced to the concept of a function. Sometimes it is easier to understand the way a function behaves if we can view it pictorially. A graph is such a pictorial form. In this unit we will first show you how to draw the graph of any given function. Then we discuss graphs of the power, exponential, logarithmic and trigonometric functions because we expect you to use these functions again and again.

As in the previous unit, the material given in this unit must be studied thoroughly, because it will be useful for the rest of the course.

Objectives

After studying this unit, you should be able to

- draw the graph of a function;
- work with the function $x \to a^x$, $a > 0$, $a = 1$;
- define and use logarithmic functions with base 2,10 and e;
- define trigonometric ratios of angles and trigonometric functions of real numbers.

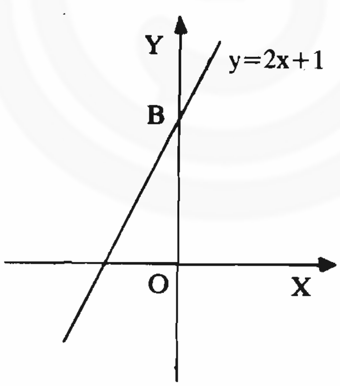


Fig. 1 : Graph of y = 2x+1

2.2 GRAPHS

Consider the function $f : \mathbf{R} \to \mathbf{R} : f(x) = 2x + 1$. Will you believe that its pictorial representation is given by the line AB in Fig. 1? You will, by the time you get to the end of this section.

To draw a graph we need to know about the Cartesian coordinate system. So let us talk about Cartesian coordinates now.

2.2.1 Coordinates

The French mathematician and philosopher, Rene Descartes, was the first to suggest a system to represent functions pictorially. It is called the Cartesian system. The basic idea behind this system is to locate points in a plane. For this, we take any point O in a plane and draw two lines passing through it and at right angles to each other (see Fig. 2). Let us name these lines X′OX and Y′OY. X′OX is horizontal, whereas Y′OY is vertical. The point O is called the **origin**. The lines X′OX and Y′OY are called **the x-axis** and **the y-axis,** respectively. OX and OY are the positive directions of the x and y axes, respectively. OX′ and OY′ are the negative directions of the x and y axis, respectively. That is, the distances of points on OX and OY from O are taken as positive. The distances of points of OX′ and OY′ from O are taken as negative.

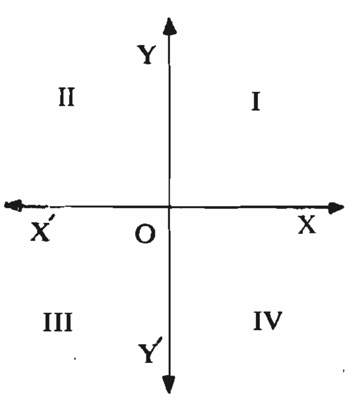


Fig. 2 : The 4 quadrants

The x-axis and the y-axis divide the whole plane into four regions, namely, the 1st, 2nd, 3rd and 4th **quadrants** (see Fig. 2).

Note that the quadrants are labelled in the anti-clockwise direction.

Now take any point P in the plane. Draw PN and PM, perpendiculars from P on the x-axis and y-axis, respectively, (see Fig. 3).

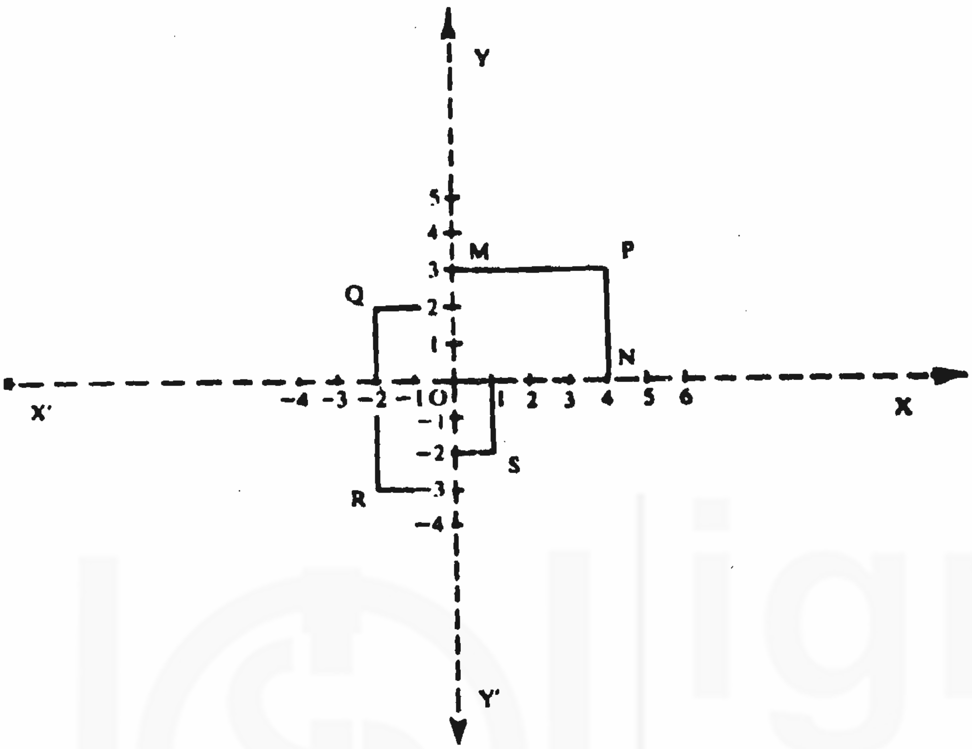


Fig. 3 : The coordinates of any point P

Then NP is the distance of the point P from the x-axis. It is called the **y-coordinate** (or **ordinate**) of P. Similarly, MP is the distance of the point P from the y-axis. It is called the **x- coordinate** (or **abscissa**) of P. The x-coordinate and y-coordinate, taken together, form the **coordinates** of P.

The coordinates of P are written as an ordered pair (MP, NP). Note that the order is important— **first the x-coordinate, and then the y-coordinate.** In Fig. 3, OM = NP and ON = MP. Thus, the point P is also given by (ON, OM).

In Fig. 3 P is given by (4,3). We use the expression P(4,3) to indicate that P's x-coordinate is 4 and y-coordinate is 3. In general we write P as P(x,y), where x and y are its x-coordinate and y-coordinate, respectively. Note that

i) $x > 0, y > 0$, if P lies in the first quadrant;

ii) $x < 0, y > 0$, if P lies in the second quadrant;

iii) $x < 0, Y < 0$, if P lies in the third quadrant;

iv) $x > 0, Y < 0$, if P lies in the fourth quadrant;

With this information you can solve the following exercise.

E1) What are the coordinates of O, Q, R and S in Fig. 3?

For any point on the x-axis, its distance from the x-axis will be zero. Hence, its y-coordinate will be zero. Similarly, the x-coordinate of any point on the y-axis is zero.

E2) What are the coordinates of M and N in Fig. 3?

Let us now get down to drawing the graph of a function.

2.2.2 How To Draw Graphs

We have just seen that finding the coordinates of a point amounts to fixing its position in the point. If we want to find the coordinates of a point in the plane, we **read** the point. Thus, in E1 and E2 you were asked to read the points, O,Q,R,S,M and N.

Now, let f be a function from A into B. We represent the members of the domain of f (i.e., A) along the x-axis, and the values of the range of f along the y-axis.

We plot all the ordered pairs $(x, f(x))$, $x \in A$, in the plane. The collection of all such points is the **graph** of f. If A is a finite set, then the graph of f consists of only a finite number of points scattered over the plane. However, if A consists of an infinite number of points, then the graph of f also consists of an infinite number of points. It is impossible to plot all these points on the graph paper. Under such circumstances, the usual technique is to plot a few points on the graph paper and then join them by a smooth free-hand curve. The curve obtained in this way is taken as the graph of the function under consideration.

Let us consider a few examples.

Example 1: Draw the graph of the functions $y = x$ and $y = x^2$. (The function $y = x^n$, $n \in \mathbf{N}$, is called the **power function** and has a lot of applications in biophysics.)

Solution: Here x is the independent variable and y is the dependent variable. First we draw the graph of the function $f : \mathbf{R} \to \mathbf{R} : f(x) = x$. For this we first draw up a table in which we obtain values of f(x) for some chosen values of x belonging to the domain of f. In the following table we find y, i.e., f(x) for $x = -3, -2, -1, 0, 1, 2, 3$.

x	−3	−2	−1	0	1	2	3
y = x	−3	−2	−1	0	1	2	3

In Fig. 4 we plot these points on a graph paper and join them to get the graph of the function $y = x$. Note that it is a straight line.

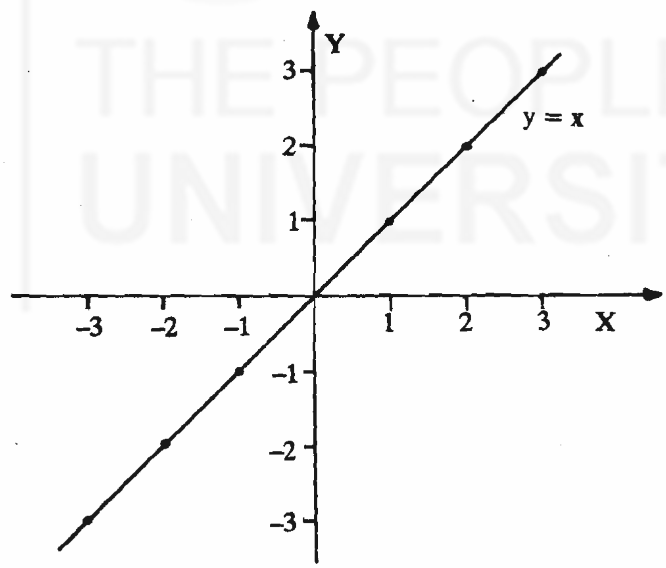


Fig. 4: Graph of y = x

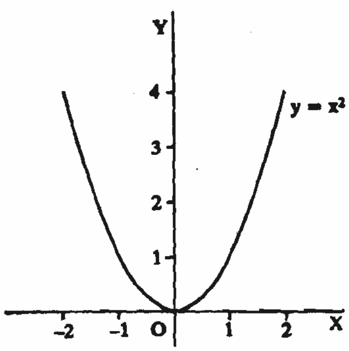


Fig. 5 : Graph of y = x²

Similarly, to obtain the graph of the function $y = x^2$ we draw up the following table.

x	−3	−2	−1	0	1	2	3
$y=x^2$	9	4	1	0	1	4	9

Then, the required figure is given in Fig. 5. This is a parabola.

Example 2: The density of an object is its mass (in grams) divided by its volume (in cubic centimetres). For different objects, each of mass 10 grams, their densities are given by the function $f : \mathbf{R}^+ \to \mathbf{R}^+ : f(V) = 10/V$. Draw the graph of this function.

Solution: The following table gives us values of $D = f(V)$ for chosen values of V.

V	.5	1	1.5	2	3
D	20	10	6.67	5	3.33

This helps us to plot the required graph (Fig. 6.).

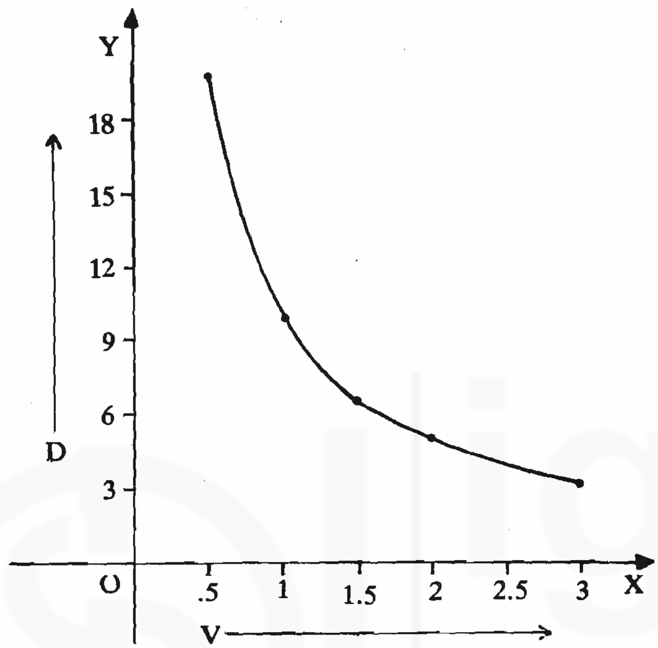


Fig. 6 : D = 10/x, v>0

Note that the gradations on the x-axis and the y-axis are different in this case. This is often done to accommodate a graph in a reasonable space.

Now you can try to draw some graphs yourself.

E3) Graph the function $y = x^3 \ \forall \ x \in [-1, 1]$.

E4) Graph the function $s = \frac{1}{2} t^2$, where $t \in \mathbf{R}^+$.

(**Hint** : s is the dependent variable and t is the independent variable.)

Now we will study two new types of functions.

2.3 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In this section we will briefly discuss two types of functions and their graphs. We first talk of exponential functions.

2.3.1 Exponential Functions

Consider the way rabbits multiply. Suppose each female rabbit gives birth to 3 female rabbits. Then there is one female rabbit in the first generation, 3 in the second, 3^2 in the third, and so on. Thus, the function $f : \mathbf{N} \to \mathbf{N}: f(x) = 3^x$ sums up this situation very well. Here $f(x)$ gives the number of female rabbits in the $(x + 1)$th generation. f is an example of an exponential function. Let us see what this means.

Definition : **An exponential function** is a function in which the unknown (or independent) variable appears as a exponent. In the example above the unknown appears as an exponent of 3.

The graph of an exponential function is called as **exponential curve.**

Example 3: Obtain the graph of the function $y = 2^t$.

Solution : As usual, we first draw up a table of values.

t	−3	−2	−1	0	1	2	3
2^t	1/8	1/4	1/2	1	2	4	8

This table suggests that the graph is as shown in Fig. 7.

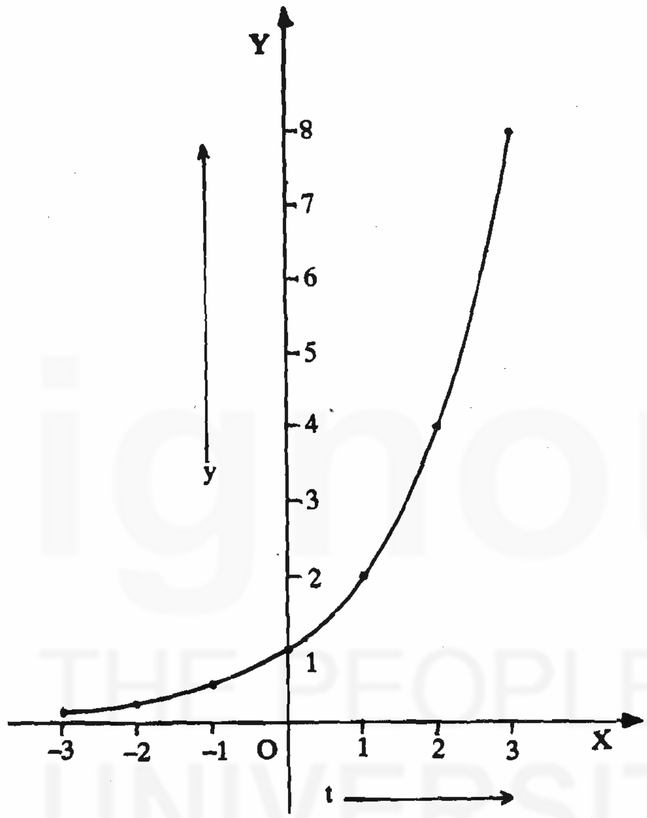


Fig. 7 : $y = 2^t$

You may like to try the following exercise now.

E5) Graph the function $y = 10^x$

The exponential function $y = e^x$ is often applied for studying growth or decay. Here e is the irrational number whose approximate rational value is 2.71828. The curve of $y = e^x$ is given in Fig. 8.

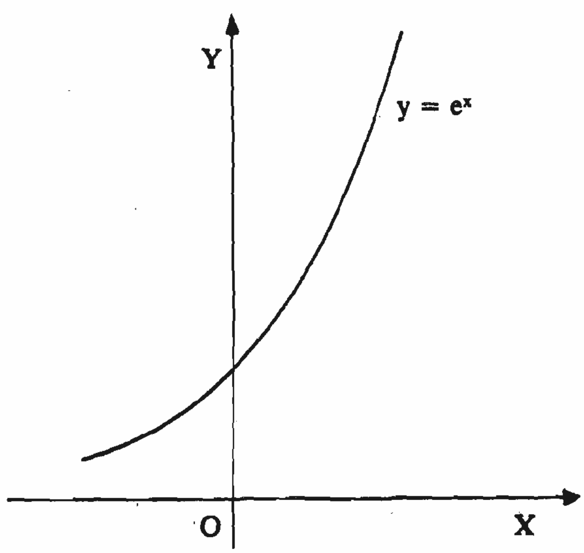


Fig. 8 : $y = e^x$

In the next example we discuss an exponential function which comes up when dealing with radioactive decay.

Example 4: A radioactive substance emits radiation continuously and, at the same time, undergoes radioactive decay. Physicists have found that radioactive substance decay is given by the function A: $\mathbf{R}^+ \mapsto \mathbf{R}^+$ defined by

$A(t) = A_0 e^{-\lambda t}$

where A_0 = amount of substance at the beginning of the decay, A(t) = amount of substance after time t has passed and λ is a constant depending upon the nature of the radioactive substance. (λ varies from one radioactive substance to another, and hence, is a **parameter**.) Find the graph of this function.

Solution: Note that λ is a constant. Once we know the substance under study. λ's value is known. Let us assume $\lambda = .301$. To obtain the different values of A we use a calculator. As t increases, $e^{-\lambda t}$ decreases, and hence, A decreases. We get the graph as in Fig. 9.

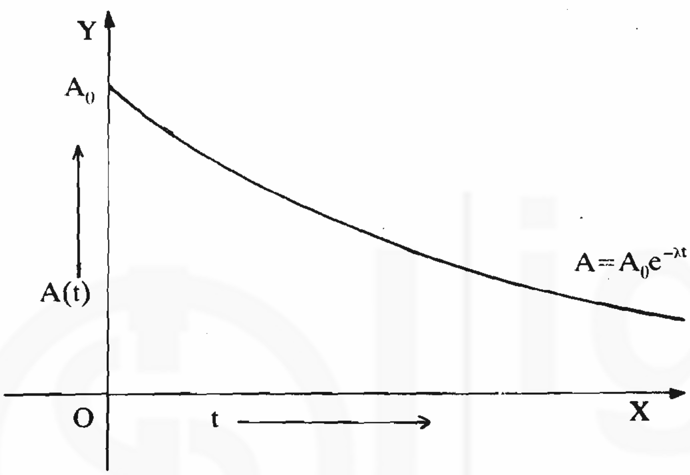


Fig. 9: Curve showing radioactive decay

Remark: Sometimes λ may not be known. In such a situation the function given above can be used to obtain λ if we can obtain the value of A(t) at a given time t_1 by observation.

Try this exercise now. Also note the difference between the graph you will get, and the graph you got in E5.

E6) Plot the graph of the function $y = 10^{-x}$.

And now we will discuss a function that is the inverse of the exponential function.

2.3.2 Logarithmic Functions

You must be familiar with logarithms, a device used to simplify calculations by "turning" complicated multiplications and divisions into additions and subtractions. Let us recall the definition of a logarithm to the base a, where a is a positive real number such that of a ≠ 1.

The **logarithm of a positive real number x, to the base a,** denoted by **$\log_a x$,** is the real number y, where $a^y = x$.

Thus, $\mathbf{y = \log_a x \Leftrightarrow a^y = x}$.

For example, $\log_3 9 = 2$, since $3^2 = 9$.

Note that $\log_a x$ is defined only for $x > 0$.

The following exercise will help you to get used to logarithms.

E 7) Calculate $\log_{10} 10$, $\log_{10} 100$, $\log_{10} 1000$.

The base a may be a rational or an irrational number. If a = 10, the logarithms are called **common logarithms,** and **we denote $\log_{10} x$ by log x.**

If a = e = 2.718282..., then the logarithms are called **natural logarithms.** They were devised by John Napier (1550 – 1617). **We denote $\log_e x$ by ln x.** Tables for calculating common, as well as natural, logarithms are easily available.

Now we make some observations about logarithms. On using the definition you can see that the following statements are true for any $a > 0$, $a \neq 1$.

i) $a^{\log_a x} = x$, for $x > 0$.

ii) $\log_a 1 = 0$,

iii) $\log_a a = 1$.

In particular, we have $\ln e = 1$, $\log 10 = 1$ and $\log_2 2 = 1$.

Logarithms to the base a obey the following three laws.

$\log_a(xy) = \log_a x + \log_a y$, $x > 0, y > 0$,

$\log_a(x/y) = \log_a x - \log_a y$, $x > 0, y > 0$,

$\log_a(x^y) = y \log_a x$, $x > 0$.

These correspond to the familiar rules of exponents, namely, $a^x \cdot a^y = a^{x+y}$, $a^x/a^y = a^{x-y}$, $(a^x)^y = a^{xy}$ $\forall$ $a,x,y \in \mathbf{R}$. Let us now define a logarithmic function.

Definition : A function $f : \mathbf{R}^+ \to \mathbf{R} : f(x) = \log_a x$, $a \neq 1$, is a **logarithmic function.** The curve of such a function is a **logarithmic curve.**

Let us look at an example.

Example 5: Graph the function $y = \log_2 x$.

Solution:

x	1/8	1/4	1/2	1	2	4	8
$\log_2 x$	− 3	− 2	− 1	0	1	2	3

leads us to the graph as given in Fig. 10.

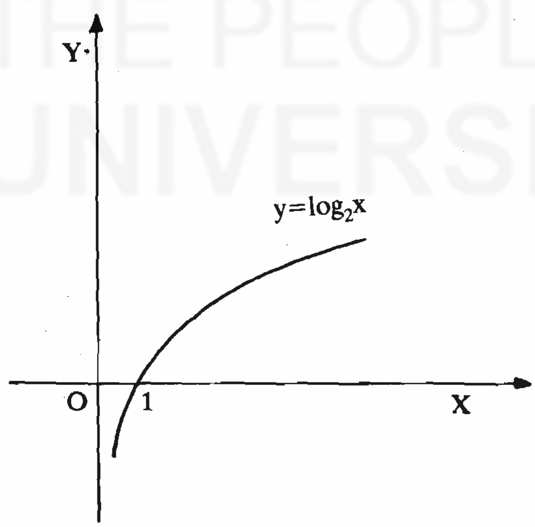


Fig. 10 : $y = \log_2 x$

Logarithms to the base 2 are also useful in situations as in Example 6.

Example 6: Each bacterium present in a culture medium, divides into two bacteria in the 2nd generation. These become 2^2 (=4) bacteria in the 3rd generation, and so on. Suppose this has given rise to a population of 4096 bacteria. How many generations did it take?

Solution: The number of bacteria in the xth generation is given by $n(x) = 2^{x-1}$. Suppose it takes x generations to reach a population of 4096 bacteria. Then $2^{x-1} = 4096$. Thus, by definition

$x - 1 = \log_2 4096 = \log_2 2^{12} = 12 \log_2 2 = 12$. Therefore, $x = 13$.

Thus, it has taken 13 generations for a single bacterium to give rise to a population of 4096 bacteria.

Try the following exercise now.

E8) Suppose there were only 3 bacteria present in a given culture medium. At the end of t generations, it was found that they have multiplied to a population of size 24576. Determine t.

Common logarithms and natural logarithms have a lot of applications. Sometimes it is convenient to convert log x to ln x and vice versa. For this we have the formula **log x = 0.4343 ln x.** If you use common logarithms for solving problems, you can use antilogarithm tables to help in obtaining a solution.

Recall that a **real number x is the antilogarithm of y, if y = log x.**

This fact is denoted by **x = antilog y.**

Thus, x = antilog y $\Leftrightarrow$ y = log x. Let us consider an example.

Example 7 : The population of a city grows exponentially. Assume that the annual rate of growth is 3.5%. What increase is expected in ten years?

Solution: In the first year the population is a, say. In the second year it will be a (1 + 3.5/100) = 1.035a, and so on. Thus, the growth in ten years is determined by the factor $(1.035)^{10}$ = y, say. To get this value we apply logarithms. We have log y = log $(1.035)^{10}$ = 10 log (1.035) = .1494, using the log tables. The antilog tables tell us that

y = antilog .1494 = 1.41. Thus, the city's population will be ay = a (1.41) = $a + \frac{41a}{100}$ after 10 years.

This means that the population will have increased by 41%.

Now Example 4 tells us that a radioactive substance decays according to the equation $A(t) = A_0e^{-\lambda t}$, that is, ln $(A_0/A) = \lambda t$, or equivalently, log $(A_0/A) = .4343\lambda t$. Now, suppose we want to find the "half-life" of a radioactive substance, that is, the time taken for A_0 to become $\frac{1}{2}A_0$. In this case, $A_0 / A = 2$.

Thus, the required time is given by .4343 λt = log 2 = .3010 (from the tables).

$\Leftrightarrow$ t = 3010/4343λ = 0.693/ λ

Thus, if λ is known for the radioactive substance, its half life can be computed. Conversely, if the half life is known, λ can be computed.

Try the following exercise now.

E 9) The population of a country is decreasing exponentially. After 1000 years it will be half of what it is at present. If it is 80 crores now, what will the population be after 100 years?

Before ending this section let us see the relationship between exponential and logarithmic functions.

In Unit 1 (Sec. 1.8) we had discussed the situation when two functions are the inverses of each other. We said that f : A $\to$ B and g : B $\to$ A are inverses of each other if f$\circ$g = I_B and g$\circ$f = I_A. This is exactly the situation when, for a > 0, a 1, f(x) = a^x $\forall$ x $\in$ **R** and g(x) = $\log_a x$ $\forall$ x $\in$ $\mathbf{R}^+$

Then we have f : **R** $\to$ $\mathbf{R}^+$ and g : $\mathbf{R}^+ \to$ **R** such that

$f(g(x)) = f(\log_a x) = a^{\log_a x}(a^x) = x \ \forall\ x > 0$, and

$g(f(x)) = g(a^x) = \log_a (a^x) = x \log_a a = x \ \forall\ x \in \mathbf{R}$.

Thus, f and g are inverse functions of each other.

Let us see how the graphs of f and g behave for a = e. In Fig. 11 we consider the graphs of f and g. They are the reflections of each other in the straight line y = x, which passes through the origin and is inclined at an angle of 45° with the positive direction of the x-axis. This means that if you fold the graph paper at the line y = x then the two graphs will coincide.

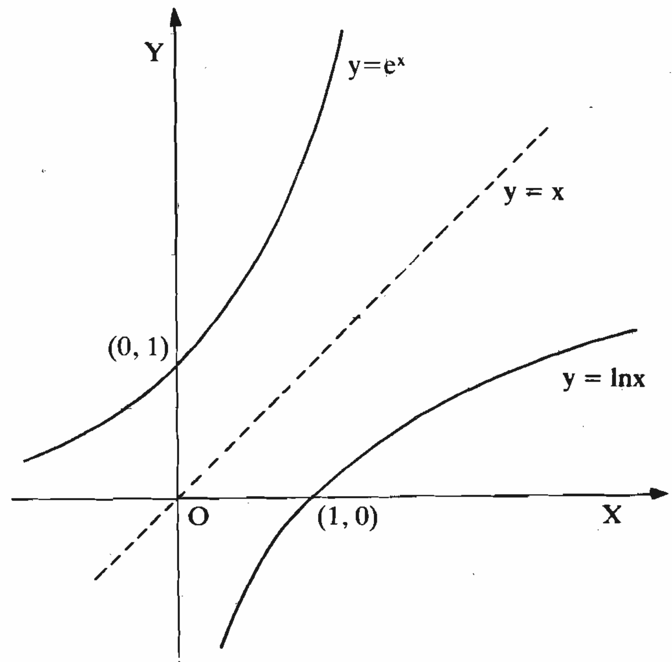


Fig. 11 : e^x and ln x are inverses of each other

E10) Draw, on the same graph paper, the graphs of the functions f and g defined by
$f(x) = 10^x \ \forall \ x \in \mathbf{R}$.
$g(x) = \log x \ \forall \ x \in \mathbf{R}^+$
Examine whether these graphs are reflections of each other in the line y = x.

And now we go to the next section, where we will study functions like sine and cosine.

2.4 SOME TRIGONOMETRY

Trigonometry is a branch of mathematics which deals with the study of triangles. The word "trigonometry" has been derived from three Greek words-tri (three), gonia (an angle) and metron (a measure). Before discussing trigonometric functions, let us recall some basic definitions.

2.4.1 Basic Definitions

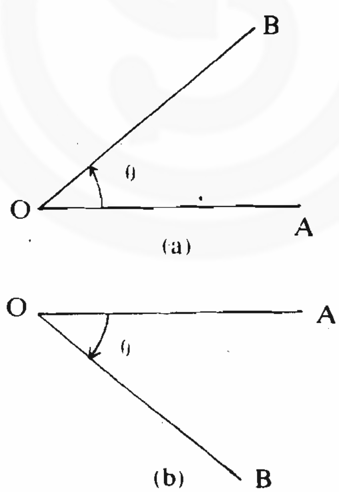
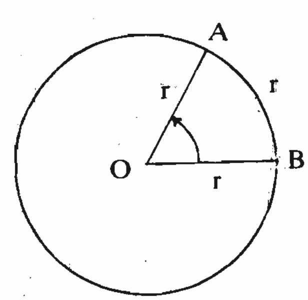


Fig. 12: Angle between two lines

Consider Fig. 12. In it you see the angle between the lines OA and OB, denoted by ∠AOB. Let us say that ∠AOB=θ. What does θ measure? It measures the magnitude of the rotation that a line undergoes while rotating from position OA to position OB. OA is called the **initial side** and OB the **terminal side** of this angle. If the rotation is in the anti-clockwise direction (Fig. 12 (a)), then θ is positive. If the rotation is clockwise (Fig. 12 (b)), then θ is negative.

There are two systems for measuring an angle. In the first system the unit for measuring an angle is a **degree**. It is defined as 1/360th of a complete revolution made in the anti-clockwise direction. Thus, according to this definition, there are 360 degrees in a complete revolution made in the anti-clockwise direction. We denote an angle's measure of θ degrees by θ°

In the second system an angle is measured in **radians.** What is a radian? Consider a circle of radius r (Fig. 13). Let its centre be O. Take any point B on the circumference of this circle.



Mark off the point A on the circle such that $\widehat{BA} = r$, where the symbol $\widehat{BA}$ denotes the length of the arc $\widehat{BA}$. (Note that the symbol $\widehat{BA}$ does not denote the length of the line segment obtained by joining B to A.) Now join the points B and A to the centre O of the circle. The angle ∠BOA, measured in the anti-clockwise direction is called one radian. What we have said is that an angle measuring one radian intercepts, on the circumference of the circle, an arc equal in length to the radius of the circle. Then, we get the formula

$$\text{measure of an angle in radians} = \frac{\text{length of arc subtending the angle}}{\text{radius of circle}}$$

Since the circumference of a circle is $2\pi r$, a line undergoing a complete revolution in the anti-clockwise direction subtends an angle of $\frac{2\pi r}{r}$ $(= 2\pi)$ radians at the centre.

Thus, 2π radians = 360°

That is. π radians = 180°

From this, it follows that

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

and

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ$$

Now you can do the following exercise.

E11) Give the value of

a) 15°, 30°, − 45° in terms of radians;

b) $\pi/3$, $\pi/2$, $(-2\pi/3)$, $3\pi/2$ in terms of degrees.

Note: In E11) (b) we have not written the word "radians" after $\pi/3$, etc. The convention that is followed is that if no unit is given after the measure of an angle, then the angle is assumed to be in radians. For example, 'an angle 1.25' means 'an angle of measure 1.25 radians'.

Now let us define some ratios related to a given angle.

2.4.2 Trigonometric Ratios

Consider the right-angled triangle ABC given in Fig. 14, where ∠ABC = θ. Then we define the following 6 terms.

The **sine of** θ (denoted by **sin** θ) = AC/AB

The **cosine of** θ (denoted by **cos** θ) = BC/AB

The **tangent of** θ (denoted by **tan** θ) = AC/BC

The **cosecant of** θ (denoted by **cosc** θ) = AB/AC

The **secant of** θ (denoted by **sec** θ) = AB/BC

The **cotangent of** θ (denoted by **cot** θ) = BC/AC

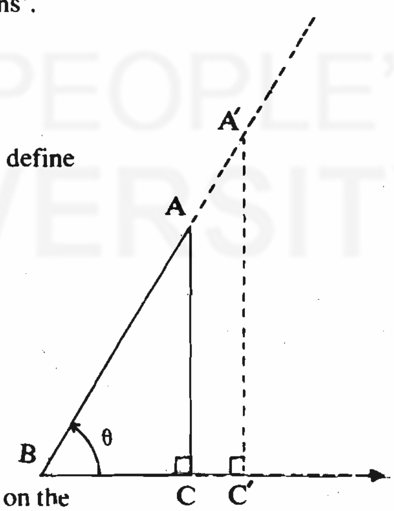


Fig. 14

These ratios appear to be dependent on A and C. But they are not. They only depend on the angle θ. To see this, consider a larger right-angled triangle A'BC', with ∠A'BC' = θ (as in Fig. 14). Then the ratios still remain the same. Why? Well because the triangles ABC and A'BC' are similar.

The six ratios defined above are called the **trigonometric ratios** of θ.

In the above case θ is an acute angle, i.e. θ < 90°. In general, given any angle θ we can obtain its trigonometric ratio as follows:

Take a line that makes an angle θ with OX, at the point O. This line will be the terminal side of the angle θ. Fig. 15 gives the 4 possible cases—when OP is in the first, second, third or fourth quadrant.

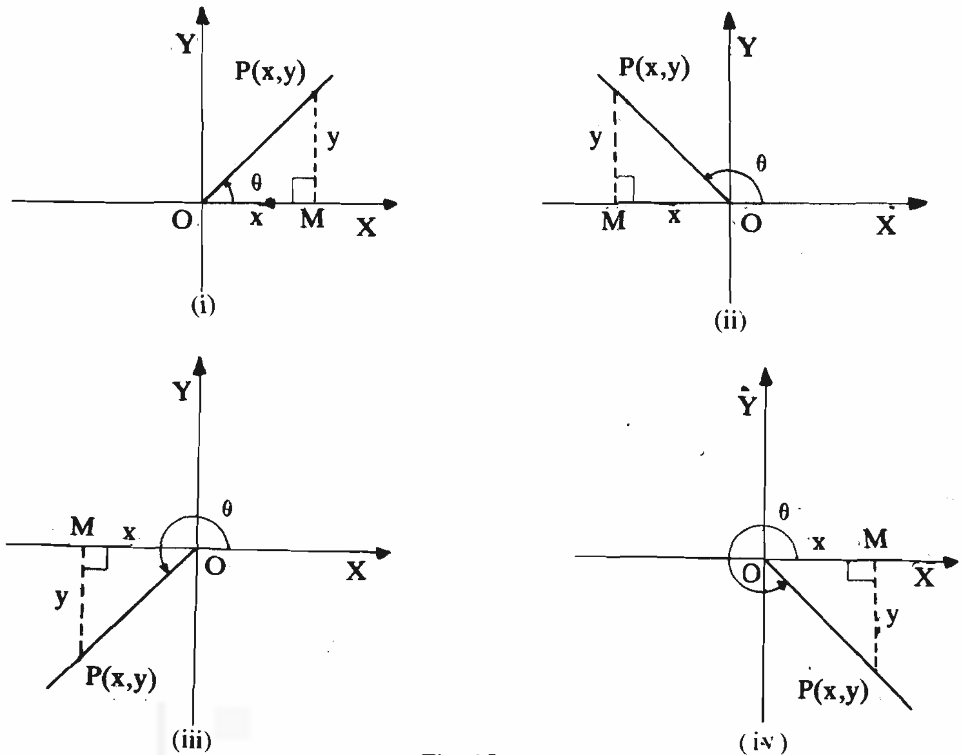


Fig. 15

Take any point P(x,y) on the terminal side and draw PM perpendicular to the x-axis. Then OM = x, PM = y. By the Pythagoras theorem OP = $\sqrt{x^2 + y^2}$ = r, say. Then we define:

sin θ = MP/OP = y/r, **cosec θ** = OP/MP = r/y,

cos θ = OM/OP = x/r, **sec θ** = OP/OM = r/x,

tan θ = MP/OM = y/x, **cot θ** = OM/MP = x/y.

If the terminal side coincides with the initial side, then θ = 0 and the point P coincides with M so that MP = 0 and OP = OM.

Hence,

sin 0 = 0, cos 0 = 1, tan 0 = 0.

cosec 0 and cot 0 are undefined.

If the terminal side coincides with OY, then M coincides with the origin O, so that MP = OP and OM = 0. In this case, θ = π/2 = 90° and sin π/2 = 1, cos π/2 = 0.

E12) Obtain sec 0, tan 90°, cot 90°, cosec 90° and sec 90°.

If the terminal side lies in the first quadrant, as in Fig. 15 (i), then the coordinates of the point P are positive. Consequently, all the six trigonometric ratios of θ are positive. Notice that here we have 0< θ < π/2. The chart in Table 1 gives the signs of the trigonometric ratios for the various cases given in Fig. 15.

Table 1 : Signs of Trigonometric Ratios of θ

Ratio	First Quadrant	Second Quadrant	Third Quadrant	Fourth Quadrant
sin, cosec	+	+	–	–
cos, sec	+	–	–	+
tan, cot	+	–	+	–

Using this table, try and solve the next exercise.

E13) Calculate the sine and cosine of π, 3π/2 and 2π.

In Table 2 we give the trigonometric ratios for certain angles.

Table 2 : Some Trigonometric Ratios

Angle	sin	cos	tan	cot	sec	cosec
$0 = 0^\circ$	0	1	0	not defined	1	not defined
$\pi/6 = 30^\circ$	1/2	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
$\pi/4 = 45^\circ$	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\pi/3 = 60^\circ$	$\sqrt{3}/2$	1/2	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$
$\pi/2 = 90^\circ$	1	0	not defined	0	not defined	1
$\pi = 180^\circ$	0	−1	0	not defined	−1	not defined
$3\pi/2 = 270^\circ$	−1	0	not defined	0	not defined	−1
$2\pi = 360^\circ$	0	1	0	not defined	1	not defined

Notice that the trigonometric ratios of 0 and 2π coincide. **In general, all the trigonometric ratios of θ and $(2\pi + \theta)$ coincide, for any angle θ.**

It is now time to point out a few interesting relationships between the trigonometric ratios. From the definition you can see that

cosec $\theta = 1/\sin\theta$, $\sec\theta = 1/\cos\theta$ and $\cot\theta = 1/\tan\theta$, for any angle θ.

Also note that $\tan\theta = \sin\theta/\cos\theta$, for any angle θ.

The discussion above tells us that knowing $\sin\theta$ and $\cos\theta$, we can get the other trigonometric ratios of θ.

We utilise this in the following important example.

Example 8: For any angle θ, $\sin(\pi/2 + \theta) = \cos\theta$ and $\cos(\pi/2 + \theta) = -\sin\theta$. Obtain the other trigonometric ratios of $\pi/2 + \theta$ in terms of those of θ.

Solution : $\tan(\frac{\pi}{2} + \theta) = \sin(\frac{\pi}{2} + \theta)/\cos(\frac{\pi}{2} + \theta) = \cos\theta\,(-\sin\theta) = -\cot\theta$.

You can similarly show that

cosec $(\pi/2 + \theta) = \sec\theta$, $\sec(\pi/2 + \theta) = -\text{cosec}\,\theta$ and $\cot(\pi/2 + \theta) = -\tan\theta$.

E14) Knowing that $\sin(\pi - \theta) = \sin\theta$ and $\cos(\pi - \theta) = -\cos\theta$, for any angle θ, obtain the trigonometric ratios of 120^0.

Now we are in a position to define and graph trigonometric functions.

2.4.3 Trigonometric Functions

Consider the function f : **R** → **R** : f(x) = sin (x radians)

This is the **sine function**. We similarly define the **cosine function** g:**R** → **R** by g(x) = cos (x radians).

Let us try to obtain the graphs of these functions. For this, note that sin x or cos x is the ratio of the length of a leg of a right-angled triangle to its hypotenuse, taken with the proper sign. Thus,

$-1 \le \sin x \le 1 \;\forall\; x \in \mathbf{R}$

and $-1 \le \cos x \le 1 \;\forall\; x \in \mathbf{R}$

Now we will obtain a table of values to help us obtain the graph of the sine function.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
sin x	0	0.5	0.866	1	0.866	0.5	0	−0.5	−0.866	−1	−0.866	−0.5	0

Then, the graph of sin x for $x \in [0,2\pi]$ is given in Fig. 16.

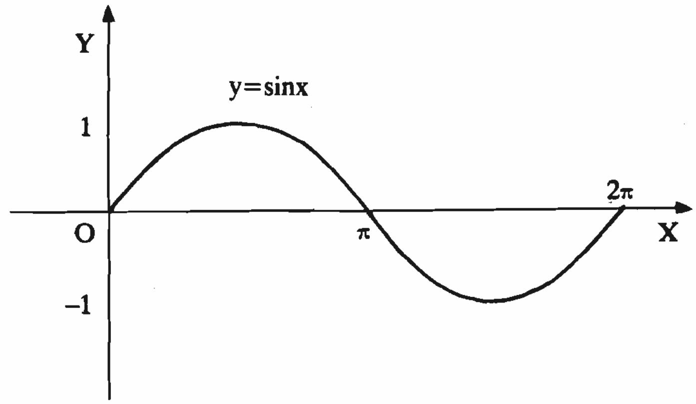


Fig. 16 : y = sin x

E15) Obtain the graph of the cosine function for $x \in [0, 2\pi]$.

The graphs of xin x and cos x, $x \in \mathbf{R}$, start repeating themselves after a gap of 2π units. Hence, their graphs over the interval $[0, 2\pi]$ will be repeated over the intervals $[2\pi, 4\pi]$, $[4\pi, 6\pi]$, ..., as well as over $[-2\pi, 0]$, $[-4\pi, -2\pi]$, ... They are examples of periodic functions, with period 2π.

A function $f:\mathbf{R} \rightarrow \mathbf{R}$ is said to be **periodic** if there exists a positive real number p such that $f(x+p) = f(x)$ for all $x \in \mathbf{R}$. p is called the **period** of f.

The trigonometric function tan x is also periodic, but its period is π, not 2π. We give its graph in Fig. 17.

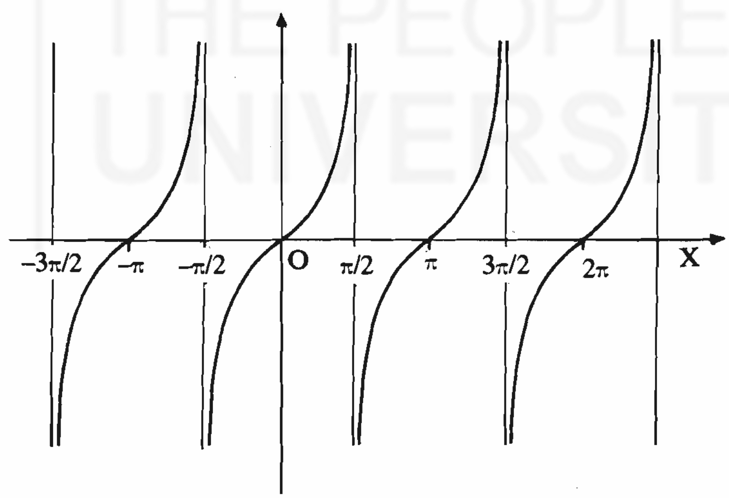


Fig. 17 : y = tan x

You will find trigonometric functions useful for fitting data concerning biological rhythms or for studying interference of light.
Let us now see what we have covered in this unit.

2.5 SUMMARY

In this unit we have discussed the following points.

1) What the graph of a function is and how to obtain it.
2) Exponential functions, along with their graphs and some applications.
3) Logarithmic functions with base 2, 10, e, their graphs and applications.

4) Trigonometric ratios of angles.

5) Trigonometric functions of real numbers, and their graphs.

2.6 SOLUTIONS/ANSWERS

E1) O (0, 0), Q(− 2, 2), R(− 2, − 3), S(1, − 2).

E2) M(0, 3), N(4, 0).

E3) We form a table of values first.

x	− 1	− 0.5	0	0.5	1	
$y = x^3$	−1	− 0.125	0	0.125	1	

Therefore, the graph will be as in Fig. 18.

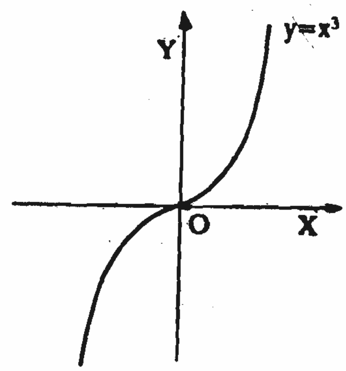


Fig. 18 : $y = x^3$

E4) The table

t	1/2	1	2	3
$s = \frac{1}{2}t^2$	$\frac{1}{8}$	$\frac{1}{2}$	2	$\frac{9}{2}$

allows us to draw the graph as in Fig. 19.

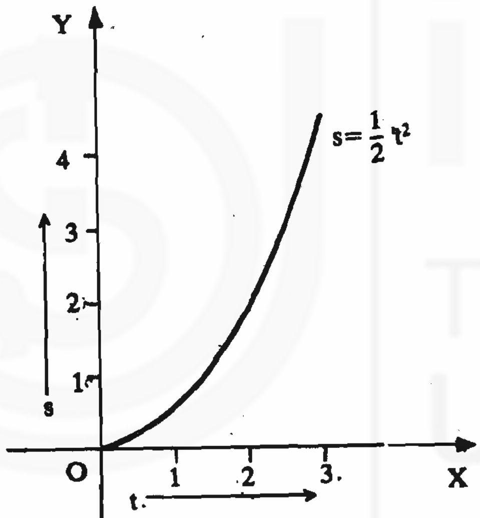


Fig. 19 : $s = (\frac{1}{2})t^2$

E5) A table of values is

x	− 1	0	1	2	3
$y=10^x$	.1	1	10	100	1000

Thus, the required graph is as shown in Fig. 20.

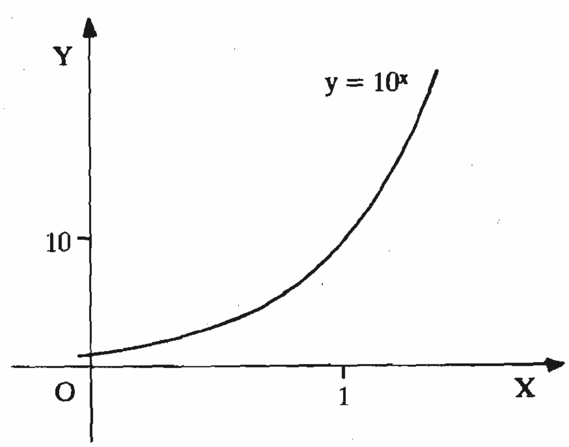


Fig. 20 : $y = 10^x$

E6) A table of values is

x	−2	−1	0	1	2
$y=10^{-x}$	100	10	1	.1	.01

Thus, the required graph is as shown in Fig.21.

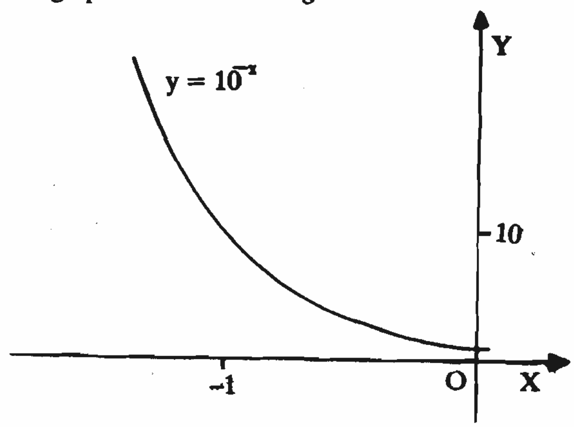


Fig. 21 : $y = 10^{-x}$

E7) $\log_{10}10=1$, $\log_{10}100=2$, $\log_{10}1000=3$.

E8) Each bacterium will give rise to 2^{t-1} bacteria at the end of t generations. ∴, 3 bacteria will give rise to $3\times2^{t-1}$ bacteria at the end of t generations. Thus,

$3\times2^{t-1} = 24576 \Rightarrow 2^{t-1} = 8192 \Rightarrow t-1 = \log_2 8192 = \log_2 2^{13} = 13 \Rightarrow t = 14$.

E9) Let the rate of population decrease be a % per year. Then after one year the population will be $80(1-a/100)$ crores, after 2 years it will be $80(1-a/100)^2$ crores, and so on. We know that $80(1-a/100)^{1000} = 40$.

Taking logarithms, we get

$1000 \log(1-a/100) = -\log 2 = -.301$

$\Rightarrow \log(1-a/100) = -0.0003 = -1+.9997$

$\Rightarrow 1-a/100 = 10^{-1}(9.99)$, using the log table

$= .999$

$\Rightarrow a = .1$

So, after 100 years the population will be $80(1-.1/100)^{100} = y$, say.

Taking logarithms we get

$\log y = \log 80 + 100\log(.999) = 1.9031 - .03 = 1.8731$

Taking antilogarithms we get $y = 74.67$ crores

E10)

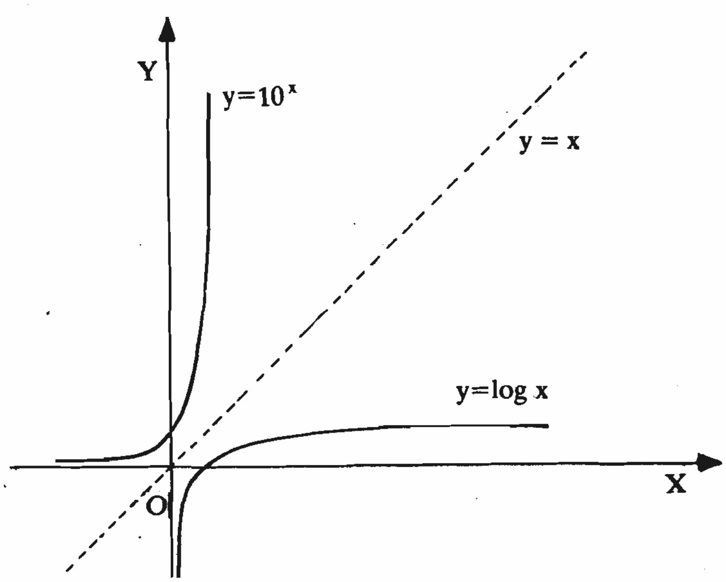


Fig. 22 : 10^x and $y = \log x$

They are reflections of each other in the line y = x.

E11) a) $15^\circ = 15 \times \pi/180$ radians $= \pi/12$ radians,
$30^\circ = \pi/6$ radians, $-45^\circ = -\pi/4$ radians.

b) $\pi/3 = (\pi/3 \times 180/\pi)^\circ = 60^\circ$
$\pi/2 = 90^\circ$, $-2\pi/3 = -120^\circ$,
$3\pi/2 = 270^\circ$.

E12) Sec 0 = 1, tan 90° is undefined, cot $90^\circ = 0$, cosec $90^\circ = 1$ and sec 90° is undefined.

E13) When $\theta = \pi$, OP = OM (in Fig. 15 (ii)), and PM = 0,

$\therefore \sin \pi = 0, \cos \pi = -1.$
When $\theta = 3\pi/2$, OM =0 (in Fig. 15(iii)), and MP = OP.

$\therefore \sin 3\pi/2 = -1, \cos 3\pi/2 = 0.$
When $\theta = 2\pi$, we use the table and Fig. 15 (iv) to get $\sin 2\pi = 0$, $\cos 2\pi = 1$.

E14) $120^\circ = 2\pi/3 = \pi - \pi/3$
Thus, using Table 2 we get
$\sin 120^\circ = \sin(\pi - \pi/3) = \sin \pi/3 = \sqrt{3}/2$.

$\cos 120^\circ = -\cos \pi/3 = -1/2$

$\tan 120^\circ = \sin 120^\circ/\cos 120^\circ = -\sqrt{3}$

cosec $120^\circ = 2/\sqrt{3}$, sec $120^\circ = -2$, cot $120^\circ = -1/\sqrt{3}$.

E15) The table of values

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
cos x	1	.866	.5	0	–.5	–.866	–1	–.866	–.5	0	.5	.866	1

suggests the graph as shown in Fig. 23.

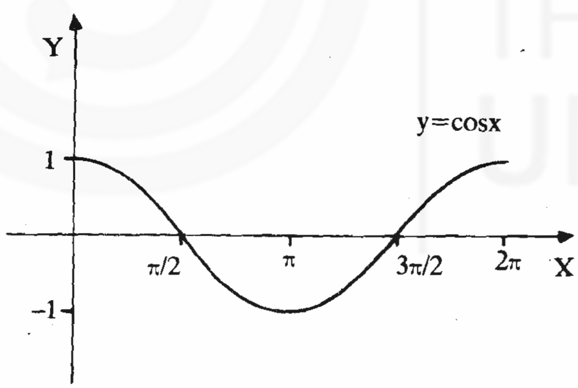


Fig. 23: y = cos x

UNIT 3 ELEMENTARY ALGEBRA

Structure

3.1 INTRODUCTION

In Unit 2 we discussed the power function x^n and its graph. x^n is an example of a polynomial. In this unit we will first discuss various polynomials and polynomial equations. In particular, we will discuss algebraic and graphical solutions of linear and quadratic equations.

We then go on to discuss various arrangements of numbers, as well as their sum, that is, sequences and series. We look at arithmetic sequences and series and geometric sequences and series in some detail.

Counting is an old activity of human beings. Sometimes actual counting is not possible or it is highly time-consuming. So, one has to devise ways of counting without actually performing the process of counting. The theory of combinations and permutations is aimed at achieving this objective. We discuss this in Section 3.4.

We end this unit with the binomial theorem.

Objectives

After reading this unit, you should be able to

- define polynomials and perform arithmetic operations on them;
- solve linear equations algebraically and graphically;
- obtain the roots of a quadratic equation;
- identify arithmetic and geometric progressions;
- define arithmetic and geometric series, find their sum upto n terms and upto infinity, wherever possible;
- evaluate the number of permutations and combinations of n objects taken r at a time, where $n > r$;
- state and use the binomial theorem.

3.2 POLYNOMIALS AND EQUATIONS

In previous units you have been exposed to various combinations of variables and constants. In this section we start with discussing algebraic expressions called polynomials.

3.2.1 Polynomials

Consider the following combinations of letters and numbers, linked together by