UNIT 1 SETS AND FUNCTIONS

Structure

1.1 INTRODUCTION

Consider any dictionary of English. It is a collection of words and their meanings. A word either belongs to this collection or not, depending on whether it is listed in the dictionary or not. This collection is an example of a set, as you will see in Section 1.2.

The concept of a set is fundamental to any mathematical study. The study of sets includes the study of operations on sets. In this unit we discuss the operations of complementation, union and intersection. We also introduce you to Venn diagrams, a pictorial way of describing sets.

Now consider the relationship $A = \pi r^2$, that links the radius r of a circle to its area A. This relationship is an example of a function. In the latter half of this unit we shall study functions in detail. You will keep using all kinds of functions throughout this course. To prepare you for this we have discussed different kinds of functions, as well as operations on functions.

Since the material covered in this unit is going to be the basis for the rest of the course, please study it carefully.

Objectives

After studying this unit you should be able to

- identify and describe a set ;
- perform the operations of complementation, union and intersection on sets;
- check whether a function is surjective, injective, bijective or monotonic;
- obtain the composite of two functions;
- perform arithmetic operations on functions.

1.2 SETS

You may have often come across categories, classes or collections of objects. In mathematics, a **set** means a **well-defined collection of objects**. The adjective 'well-defined' means that given an object we should be able to decide whether it belongs to the collection or not. For example, the collection of all chemical elements is a set. This is because any object is either a chemical element or not, and accordingly it does or does not belong to the collection. On the other hand, the collection of all intelligent human beings is not a set. Why? Because, given a human being, one person may feel that she or he is intelligent while another may not. So, the collection is not well-defined.

Now we give some more examples of sets which you may often come across.

i) The set of all natural numbers, denoted by **N**.

ii) The set of all integers, denoted by **Z**.

iii) The set of all rational numbers, denoted by **Q**.

iv) The set of all real numbers, denoted by **R**.

v) The set of all plants which produce O_2. (All such plants contain chlorophyll.)

vi) The set of all organic compounds.

You may like to try this exercise now.

E1) From the collections mentioned below, identify the sets from the mathematician's point of view:

a) The collection of all good people in India.

b) The collection of all those people who have been to Mars.

c) The collection of prime numbers.

d) The collection of all even natural numbers.

A **prime number** is a natural Number other than one, whose only factors are one and itself.

An object that belongs to a set is called an **element** or **member** of that set. For example, the hyena is an element of the set of all carnivorous animals.

It is customary to use capital letters A, B, C, etc., to denote sets. The small letters a, b, c, x, y, etc., are usually used to denote their elements.

The symbol $\in$ stands for 'belongs to'. It was suggested by the Italian mathematician Peano (1858-1932).

We symbolically write the statement 'a is an element of the set A' as $\mathbf{a \in A}$.

If a is not an element of A or, equivalently, a does not belong to A, we write it as $\mathbf{a \notin A}$.

So, for example, let A be the set of all carnivorous animals, h denote a hyena and e denote an elephant. Then $h \in A$ and $e \notin A$.

Try the following exercise now.

E2) Which of the following statements are true?

a) $2 \in \mathbf{N}$

b) $2 \notin \mathbf{N}$

c) $\sqrt{2} \in \mathbf{R}$

d) $\sqrt{2} \in \mathbf{Q}$

Now, you know that a number is either rational or irrational. So, what will the set of all numbers that are both rational and irrational be? It will not have any element.

A set which has no element is called an **empty set** (or a **void set,** or a **null set**). It is denoted by the Greek letter ϕ.

A set which has at least one element is called a **non-empty set**. It is customary to describe a non-empty set by writing down all its members within curly brackets. For instance, the set of all the factors of 10 is $\{1, 2, 5, 10\}$.

Sometimes a set may have too many elements to be able to write them all down. In this case we can describe the set in two ways — the **listing method** and the **property method**.

In the first method we list some of the elements of the set, enough to exhibit some pattern which its elements follow. For example, the set N of natural numbers can be described as

$\mathbf{N} = \{1,2, 3,......\}$.

This method of representing sets is called the **listing method** (or tabular method, or roster method).

In the second method of describing a set we describe its elements by means of a property possessed by all of them. As an example, consider the set S of all patients suffering from cancer. This set S can be written in the form

S = {x : x is a patient suffering from cancer}

' stands for 'such that'.

This method of describing a set is called the **property method** or the set-builder method.

In some cases we can use either method to describe the set under consideration. For instance, the set E of all natural numbers less than 10 can be described as

E = { 1 ,2 ,3, 4, 5, 6, 7, 8, 9} (by the listing method) or

E = {x : x is a natural number less than 10} (by the property method).

Before going further, we would like to make an important remark.

Remark : The set {1, 2, 3, 3} is the same as {1, 2, 3}. That is, while listing the elements in a set we do not repeat them.

You will be able to do the following exercises now.

E3) Describe the following sets by the listing method.

a) {x : x is the smallest prime number}

b) {x : x is a divisor of 12}

c) $\{x : x \in \mathbf{Z} \text{ and } x^2 = 4\}$

d) $\{x : 3x - 5 = 19\}$

e) the set of all letters in the word 'malayalam'

The set in (c) can also be written as $\{x \in \mathbf{Z} : x^2 = 4\}$.

E4) Describe the following sets by the property method.

a) {1, 4, 9, 16,...}

b) { 2, 3, 5, 7, 11, 13, 17, ...}

c) {....., –6, –4, –2, 0, 2, 4, 6,...}

E5) Give an example of a non-empty set which can be represented only by the property method.

While solving E3 you have come across sets consisting of exactly one element. Such a set is called a **singleton**. The singleton containing x is usually written as {x}.

Remark : The element x is not the same as the set {x}. In fact, $x \in \{x\}$.

A set, which has a finite number of elements, is called a **finite set**. By convention, the empty set is considered to be a finite set.

A set which is not finite is called an **infinite set**.

Some examples of infinite sets are **N, Q, R** and the set of points on a given line.

An important set that you will be meeting again and again is an **interval**. For any two real numbers a and b, a < b, we define the following sets of elements of **R**:

i) the **open interval** $]a,b[= \{x \in \mathbf{R} : a < x < b\}$

ii) the **closed interval** $[a,b] = \{x \in \mathbf{R} : a \leq x \leq b\}$

iii) the **semi-closed intervals** $[a,b[= \{x \in \mathbf{R} : a \leq x < b\}$ and $]a,b] = \{x \in \mathbf{R} : a < x \leq b\}$

The following exercise will help you in getting used to the notion of finite and infinite sets.

E6) Which of the following sets are finite, and which are infinite?

a) **Z**, b) ϕ,

c) The set of points on the circumference of a circle,

d)]0, 1[, e) [–1, 1], f) {1, 2,....,100}.

Let us now see how we can compare two sets.

1.3 EQUALITY OF SETS

We know that the set **N** is the same as the set {1, 2, 3,....}. To understand what we mean by two sets being the same, let us first see what we mean by a subset.

Consider two sets A and B, where

A = set of all patients suffering from cancer,

B = set of all patients suffering from blood cancer.

Each element of B is also an element of A. In such a case we say that B is a subset of A.

In general, we have the following definition.
Definition: A set X is a **subset** of a set Y if every member of X belongs to Y. This is denoted by $X \subseteq Y$. We also say X **is contained in** Y.

In the same circumstances, we say Y **contains** X, and denote it by $Y \supseteq X$.

'∃' denotes 'there exists'.

Now, there are cancer patients who do not have blood cancer. That is, $\exists\, x \in A$ such that $x \notin B$. In this case we say that B is a proper subset of A. In general, X is a **proper subset** of Y if X is a subset of Y and $\exists\, y \in Y$ such that $y \notin X$. This is denoted by $X \subset Y$.

Note that, for any set A, $A \subseteq A$, since every element of A is in A. In other words, every set is a subset of itself. But A is **not** a proper subset of itself.

Also, for any **set** A, $\phi \subseteq A$.

Try the following exercises now.

E7) Write down all the subsets of {1, 2, 3}. How many of these contain
   a) no element, b) one element, c) two elements, d) three elements?

'⇒' denotes 'implies'.

E8) Show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (**Hint** : Show that $a \in A \Rightarrow a \in C$.)

Now, let us look at the sets
A = set of even natural numbers less than 10, and

B = {2, 4 ,6, 8}.

Every member of A is a member of B. Therefore, $A \subseteq B$. Similarly, $B \subseteq A$. In such a case we say that A is equal to B.

In general, we have the following definition.
Definition : Two sets A and B are called **equal** if $A \subseteq B$ and $B \subseteq A$. This is symbolically written as **A = B**.

Let us look at a few examples.

Example 1: What is the relationship between the sets

i) {1, 2, 3} and {2, 1, 3} ?

ii) $\{x : x + 1 = 3\}$ and {2} ?

Solution : i) Since every element of the first set belongs to the second and vice versa, both sets are equal.

(Note that this example shows that changing the order in which the elements are listed does not alter the set.)

ii) Both are equal.

Now, for an interesting exercise!

E9) If A = B and $B \supseteq C$, what is the relationship between A and C?

Let us now look at sets pictorially.

1.4 VENN DIAGRAMS

Sets and their relationship can be represented very simply and pleasantly by diagrams, called **Venn diagrams**. An English logician, John Venn (1834–1923), invented them. In these diagrams sets of any kind are graphically represented by sets of points. To be able to draw a Venn diagram you would need to know what a universal set is.

In any situation involving two or more sets, we first look for a convenient large set which contains all the sets under discussion. This large set is called the **universal set**, and is denoted by U. For example, if we are talking about the set of women directors and the set of women scientists, then we can take our universal set to be the set of all women.

Again, if we wish to talk about the sets of integers and rational numbers, we could take the set of rational numbers or the set of real numbers as our universal set.

Now suppose we are discussing three sets A, B and C. We choose our universal set U. The

Venn diagram of such a situation would be shown as follows: A rectangle represents U. The subsets A, B and C are represented by closed regions lying completely within the rectangle. These regions may be in the form of a circle, ellipse or any other shape. To clarify what we have just said, consider the following example.

Example 2: Draw a Venn diagram to represent the sets

$U = \{1, 2, ..., 10\}$, $A = \{1, 2, 3\}$ $B = \{3, 4, 5\}$, $C = \{6, 7\}$.

Solution : See Fig.1.

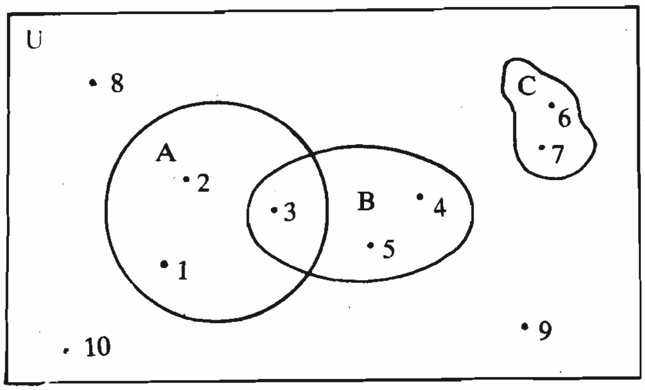

Fig.1

We have denoted A by a circle, B by an ellipse which intersects A and C as another closed region. The points 8, 9 and 10 don't lie in any of A, B or C.

Note that 3 belongs to both A and B. Therefore, it lies in the circle as well as the ellipse. Also note that A and C do not have any elements in common. Therefore, the regions representing them do not cut each other. For the same reason the regions representing B and C do not cut each other.

Of course, we could have drawn all three sets in the shape of circles.

What happens if $A \subset B$, that is, A is a proper subset of B? Well then, we can just take B to be our universal set. The Venn diagram in Fig.2 represents this situation.

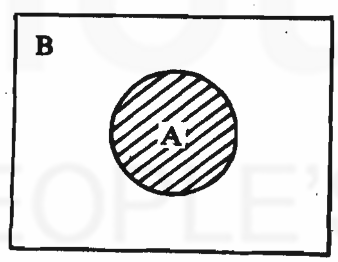

Fig. 2

Try this exercise now.

E10) How would you represent the following situation by a Venn diagram?
The set of all rectangles, the set of all squares and the set of all parallelograms.

Now that you are familiar with Venn diagrams, let us discuss the various operations on sets.

1.5 OPERATIONS ON SETS

You must be familiar with the basic operations on real numbers — addition, subtraction, multiplication and division. In using these we combine two real numbers at a time in different ways, to obtain another real number. Similarly, we can obtain new sets by applying certain operations to two or more given sets. In this section we shall discuss the operations of complementation, intersection and union.

We start with finding the complement of a set.

1.5.1 Complementation

Consider the sets **N** and [0, 1]. There are elements of **N** that do not belong to [0, 1], like 2, 3, etc. The set of these elements is the complement of [0, 1] in **N**.

We have the following definition.

Definition : Let A and B be two sets. The **complement of A in B**, denoted by **B ~ A,** is the set $\{ X \in B : x \notin A\}$.

Similarly, $A \sim B = \{ x \in A : x \notin B\}$.

If B is the universal set U, then B ~ A is U ~ A. This set is called **the complement of the set A**, and is denoted by A′.

The unshaded area in Fig.2 denotes the set B ~ A (or A′, since B = U in this case). This diagram shows us theat x ∈ A′ **if and only if** x ∉ A.

E11) a) Represent the following sets in a Venn diagram: The set P of all prime numbers, the set **Z** and the set **Q** ~ **Z**.

b) Is the set **Z** ~ P finite or infinite?

The next operation we discuss is that of the intersection of two or more sets.

1.5.2 Intersection

Let A and B be two subsets of a universal set U. The **intersection** of A and B will be the set of points that belong to both A and B. This is denoted by **A ∩ B**.
Thus, A ∩ B = { x ∈ U : x ∈ A and x ∈ B}.

To clarify this idea consider the following example.
Example 3: Consider the people whose blood group is MN. Each person has the antigen M or the antigen N. Let S be the set of people with antigen M and T the set of people with antigen N. What is S ∩ T?
Solution : S ∩ T = set of people with both antigens M and N.

We would like to make some remarks at this point.
Remarks : i) For any two sets A and B, **A ∩ B ⊆ A** and **A ∩ B ⊆ B.**

ii) If A ⊆ B, then A ∩ B = A.

iii) A ∩ A = A and A ∩ φ = φ, for any set A.

iv) For any two sets A and B, A **~B = A ∩ B′**.

An observation that you may have already made is that, for any two sets A and B, A ∩ B = B ∩ A. This is because the elements which are common to both A and B are also the elements which are common to both B and A! This shows that the intersection of sets is a commutative operation.

Of course, there do exist pairs of sets that have no element in common. For example, for any set A, A and A′ have no element in common. This leads us to the following definition.
Definition : If A and B have no common element, we say that they are **mutually disjoint,** or **disjoint**, and write this as A ∩ B = φ.

Now let us represent the intersection of sets by means of Venn diagrams. The shaded region in Fig. 3 represents the set A ∩ C, which can be seen to be non-empty. But A and B have no

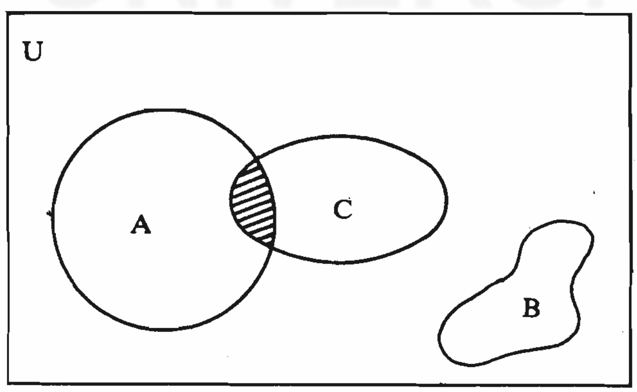

Fig.3

common element, that is A ∩ B = φ. From this diagram we can also see that neither is A ⊆ Cnor is C ⊆ A. Both C ~ A and A ~ C are non-empty sets. See how much information a Venn diagram can convey!
What situation does Fig. 4 represent? It shows two sets A and B with A ⊂ B, that is, A is a proper subset of B. Then the shaded area shows A ∩ B = A.

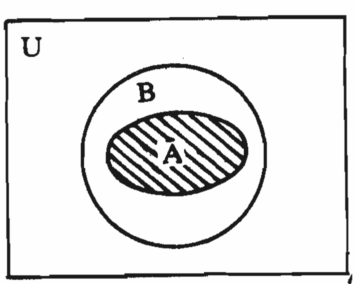

Fig.4

Try the following exercises now.

E12) If U = {1, 2, 3,...}, A = {1, 3, 5,...}, B = {2, 4, 6,...}, C = {2, 3} and D = {1}, find A ∩ B, B∩ C, C ∩D and B ∩ D.

E13) Let C and H be the sets of carnivorous and herbivorous animal species. Take U to be set of all animals species. Represent H, C, C′, C ~ H and H ~ C in a Venn diagram. (Note that $C \cap H \neq \phi$.)

Just as we have obtained the intersection of two sets. We can define the intersection of 3, 4 or more sets.

Definition : The intersection of n sets A_1, A_2, ..., A_n is defined to be the set $\{ x : x \in A$ for every $i = 1, ..., n \}$. This is denoted by $A_1 \cap A_2 \cap A_3 \cap .... \cap A_n$ or

$$\bigcap_{i=1}^{n} A_i$$

Let us look at an example involving the intersection of 3 sets.

Example 4: The patients $P_1, P_2, ..., P_{10}$ were examined by a doctor in a dispensary. She found that P_1, P_2, P_4 and P_6 suffered from viral fever; P_2, P_3, P_7, P_8 and P_{10} had diarrhoea; and all of them were anaemic. Which of them had viral fever, diarrhoea and anaemia?

Solution : To find a suitable answer to this question, let

V = set of patients with viral fever = $\{P_1, P_2, P_4, P_6\}$,

D = set of patients with diarrhoea = $\{P_2, P_3, P_7, P_8, P_{10}\}$

A = set of patients with anaemia = $\{P_1, P_2, ..., P_{10}\}$

We want to find $V \cap D \cap A$. We first find $V \cap D$ and then take its intersection with A. Or, we can first find $D \cap A$ and then take its intersection with V. In either case we'll get the same answer, namely, $\{P_2\}$.

Try the following exercise now.

E14) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $C = \{1, 4, 7, 8\}$.

Determine $A \cap B \cap C$. Also verify that

a) $A \cap B \cap C = (A \cap B) \cap C$

b) $A \cap B \cap C = A \cap (B \cap C)$

The operation of intersection of sets is associative.

Let us now look at the final operation on sets that we will discuss.

1.5.3 Union

Suppose we have two sets $A = \{x \in \mathbf{R} : x < 10\}$ and $B = \{x \in \mathbf{R} : x \geq 10\}$. Then any element of **R** belongs either to A or to B, because any real number will be either less than 10 or greater than equal to 10. In this case we will say that **R** is the union of A and B. In general, we have the following definition.

Definition : Let A and B be two sets. The set of all those elements, which belong to A or to B or to both A and B, is called the **union** of A and B. It is symbolically written as $\mathbf{A \cup B}$. Thus

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

Before going further we make some remarks.

Remarks: i) Since $A \cup B$ contains all the elements of A as well as B, it follows that

$A \subseteq A \cup B$, $B \subseteq A \cup B$.

In fact, $A \cap B \subseteq A \subseteq A \cup B$, $A \cap B \subseteq B \cup A$.

ii) For any set A, $\mathbf{A \cup A = A}$ **and** $\mathbf{A \cup \phi = A}$.

Now let us look at an example.

Example 5: Find $\mathbf{N} \cup \mathbf{Z}$.

Solution : Now $\mathbf{N} = \{1, 2, 3, ...\}$ and $\mathbf{Z} = \{... -3, -2 -1, 0, 1, 2, ...\}$. We want to find $\mathbf{N} \cup \mathbf{Z} = \{x : x \in \mathbf{N}$ or $x \in \mathbf{Z}\}$. Note that $\mathbf{N} \subseteq \mathbf{Z}$.

Thus, $x \in \mathbf{N} \Rightarrow x \in \mathbf{Z}\}$. This immediately tells us that

$\mathbf{N} \cup \mathbf{Z} = \{x : x = \in \mathbf{Z}\} = \mathbf{Z}$.

Example 5 is a particular case of the general fact that $\mathbf{A \subseteq B \Rightarrow A \cup B = B}$.

Try these simple exercises now.

E15) Show that, for any two sets A and B, $A \cup B = B \cup A$, that is, the operation of union is commutative.

E16) If A is the set of red-haired people and B the set of black-haired people, determine

$A \cup B$ and $A \cap B$.

It is easy to visualise unions of sets by Venn diagrams. Consider Fig.5. In this diagram we see four sets A, B, C and D, and the universal set U.

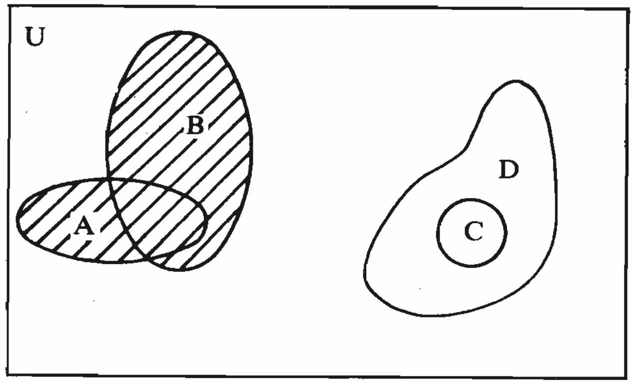

Fig. 5

The shaded area represents $A \cup B$. This area, along with the area enclosed by D, represents $A \cup B \cup D$. $C \cup D$ is the area enclosed by both C and D, which is just D, since $C \subseteq D$. Now for some small exercises.

E 17) Let U denotes the set of all human beings. A the set of individuals suffering from typhoid and B the set of individuals suffering from cholera. Interpret $A \cup B$.

E 18) Let U be the whole real line **R**, $A = \{x \in \mathbf{R} : 0 \le x \le 1\}$ and $B = \{x \in \mathbf{R} : 1 \le x < 3\}$. Determine $A \cup B$.

E 19) What can you say about A and B if $A \cup B = \phi$?

Just as we have defined the intersection of several sets, we can define the union of several sets.

Definition : The union of n sets $A_1, A_2, \ldots, A_n$ is the set $\{x : x \in A_i \text{ for some } i, 1 \le i \le n\}$.

This is denoted by $A_1 \cup A_2 \cup \ldots \cup A_n$ or $\bigcup_{i=1}^{n} A_i$.

The operation of union of sets is associative.

Suppose we want to calculate the union of 3 sets, A, B and C. Then we can first find $A \cup B$ and then find the union of this set with C; or we can first find $B \cup C$, and then find the union of A with this set. That is, $(A \cup B) \cup C = A \cup (B \cup C)$. The following exercise is about this fact.

E 20) If $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$, $C = \{1\}$, determine $A \cup B \cup C$. Verify that $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$.

The next exercise is a particular case of the general fact that for any two sets A and B, $A \cup B = (A \sim B) \cup (A \cap B) \cup (B \sim A)$. This can be clearly seen in the Venn diagram (Fig.6).

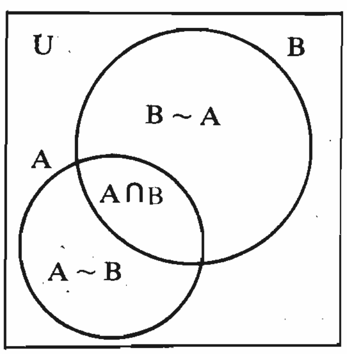

Fig.6

E 21) Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6\}$. Verify that $A \cup B = (A \sim B) \cup (A \cap B) \cup (B \sim A)$.

Now let us see the relationships between the various operations.

1.5.4 Distributive Laws, De Morgan's laws

You must be familiar with the distributive law that connects the operations of multiplication and addition of real numbers. It is

$a \times (b + c) = a \times b + a \times c$, $a, b, c \in \mathbf{R}$.

Similarly, we have two distributive laws that relate the union and intersection of sets. They are

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \ldots\ldots (1)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \ldots\ldots (2)$$

for any three sets A, B and C. In (1), $\cap$ distributes over $\cup$ and in (2), $\cup$ distributes over $\cap$.

We will not prove them, but will illustrate their validity by some examples and exercises.

Example 6: Verify the distributive laws for the sets **N**, **Q** and **R** in place of A, B and C.

Solution : We first show that $\mathbf{N} \cap (\mathbf{Q} \cup \mathbf{R}) = (\mathbf{N} \cap \mathbf{Q}) \cup (\mathbf{N} \cap \mathbf{R})$.
Now $\mathbf{Q} \cup \mathbf{R} = \mathbf{R}$, since $\mathbf{Q} \subseteq \mathbf{R}$.
Therefore, $\mathbf{N} \cap (\mathbf{Q} \text{ U } \mathbf{R}) = \mathbf{N} \cap \mathbf{R} = \mathbf{N}$, sinec $\mathbf{N} \subseteq \mathbf{R}$
Also $\mathbf{N} \cap \mathbf{Q} = \mathbf{N}$ and $\mathbf{N} \cap \mathbf{R} = \mathbf{N}$.
Therefore, $(\mathbf{N} \cap \mathbf{Q}) \cup (\mathbf{N} \cap \mathbf{R}). = \mathbf{N} \cup \mathbf{N} = \mathbf{N}$.
Thus, $\mathbf{N} \cap (\mathbf{Q} \cup \mathbf{R}) = (\mathbf{N} \cap \mathbf{Q}) \cup (\mathbf{N} \cap \mathbf{R})$.

Now, to verify that $\mathbf{N} \cup (\mathbf{Q} \cap \mathbf{R}) = (\mathbf{N} \cup \mathbf{Q}) \cap (\mathbf{N} \cup \mathbf{R})$ note that both sides are equal to **Q**. Hence the law holds.

Now try the following exercise.

E 22) Let A = {1}, B = {2, 3, 4}, C = {3, 4, 5}. Verify that

a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Now we state two more laws that relate the operation of finding the complement of a set to that of the intersection or union of sets. These are known as De Morgan's laws, after the British mathematician Augustus De Morgan (1806-1871).
They state that

$(A \cap B)' = A' \cup B'$ .... (3)

$(A \cup B)' = A' \cap B'$ .... (4)

for any two subsets A and B of a universal set U.

Let us verify these laws for the following examples.
Example 7: Let U = {1, 2, 3, 4}, A = {1, 2} and B = {2, 3, 4}. Verify De Morgan's laws.
Solution : Now A' = {3, 4}, B' = {1}, so that $A' \cup B'$ = {1, 3, 4}.
Also $A \cap B$ = {2}, so that $(A \cap B)'$ = {1, 3, 4}. Thus, $(A \cap B)' = A' \cup B'$
Now $A' \cap B' = \phi$. Also $A \cup B$ = {1, 2, 3, 4} = U, so that
$(A \cup B)' = U' = \phi$. Therefore, $(A \cup B)' = A' \cap B'$.

Try this exercise now.

E 23) Let U = {1, 2, 3, 4,...}, A = {2, 4, 6,...} and B = {1, 3, 5,...} Verify De Morgan's laws in this case.

Now that we have discussed sets let us talk about functions.

1.6 FUNCTIONS

In this section we discuss some rules or relations that link members of two given sets. But first we recall what variables and constants are.
Definition : A variable is a symbol which can take different values in the problem under consideration. A **constant** is a symbol which takes a fixed value throughout the problem under consideration.

For example, the rule $s = \frac{1}{2}gt^2$ gives the distance, in centimetres, that a freely falling object covers in t seconds, g being 980 cm/sec^2. In this case s and t are variables, since they can take any real value. g is a constant because it is a symbol representing a fixed number.

Note that the value of s depends on the value of t. In this case we say that s is a **dependent variable** and t is an **independent variable.**

Another example is the relationship between the radius r and the area A of a circle. This is given by the rule $A = \pi r^2$, where A and r are the variables and the irrational number π is a constant. We can find the area of a circle only after knowing its radius. So, r is an independent variable and A depends on r.

The rules that we have given in the examples above are examples of functions. We now define a function.
Definition : Let A and B be two non-empty sets. A **function** from A into B is a rule which assigns to each element of the set A exactly one element of the set B. Thus,
i) every member of A is linked to some member of B.
ii) every member of A is linked to **only one** member of B.

An example of a function that you all must be familiar with is the relation between the set of fingerprints of human beings and the set of humans. Each fingerprint x is associated to one and only one human being y.

A function is usually denoted by the letters f, g, h, etc. We denote the fact that f is a function from A into B by $F : A \to B$.
If f assigns $y \in B$ to $x \in A$, we say that y is the **image** of x under f and write $y = f(x)$.

If $f : A \to B$ is a function, then A is called the **domain** of f, and B is called the **codomain** of f. The domain and codomain may be any non-empty set, finite or infinite.

If the codomain of f is **R**, f is called a **real valued function**. For example, the rule $A = \pi r^2$, that we have just mentioned, is a function from **R** to **R**. Thus, it is a real valued function whose domain is also **R**.

We will usually be dealing with real valued functions in the next two blocks of this course.

A function can be denoted in several ways. For example, the function $f : \mathbf{N} \to \mathbf{N}$, which relates each natural number n to n + 1, can be written briefly as $f : \mathbf{N} \to \mathbf{N} : f(n) = n + 1$ or $f : \mathbf{N} \to \mathbf{N} : n \mapsto n + 1$. Similarly, the function given by $s = \frac{1}{2} gt^2$ can be described as $f : \mathbf{R} \to \mathbf{R} : f(t) = \frac{1}{2} gt^2$.

Now consider the following example.
Example 8 : Which of the following rules are functions?

i) $f : \mathbf{R} \to \mathbf{R} : f(x) = x$.

ii) $g : \{1, 2\} \to \{3,7\} : g(1) = 3, g(2) = 3$.

iii) $h : \{1, 2\} \to \{3,7\} : h(1) = 3, h(1) = 7, h(2) = 7$.

Solution : i) f is a function since it relates each real number to exactly one real number.
ii) g is also a function.

iii) h is not a function since it relates 1 to two elements of the codomain.

The function given in (i) above is the identity function on **R**.
In fact, for any set A, we can define the **identity function**

' $\forall$ ' denotes 'for all'

$I_A : A \to A$ by $I_A(x) = x \;\forall\; x \in A$.

When we are given a function we obtain another set that is related to it, namely, its range.
Definition : Let $f : A \to B$ be a function. The set $R_f = \{f(x) : x \in A\}$ is called the **range** of f.

Since $f(x) \in B$ for each $x \in A$, it is clear that R_f is a subset of B, that is, $R_f \subseteq B$.

In Example 8, $R_f = \{x : x \in R\} = \mathbf{R}$ = the codomain of f. Also $R_g = \{g(1), g(2)\} = \{3\}$, a proper subset of the codomain.

Let us look at some examples now.

$]1, \infty[= \{x \in \mathbf{R} : 1 \le x\}$

Example 9: Consider $f : \mathbf{R} \to \mathbf{R} : f(x) = x^2 + 1$. Find its domain, codomain and range.
Solution : Since $x^2 \ge 0$ for all $x \in \mathbf{R}$, we have $R_f = [1,\infty[$, a proper subset of **R**, the codomain of f. The domain of f is **R**.
Example 10 : Define $f : A \to B : f(x) = b$, where b is a fixed element of B. find R_f
Solution : Since $f(x) = b \;\forall\; x \in A$, we get $R_f = \{b\}$.

Example 10 leads us to some definitions.
Definition : A function whose range consists of only one element is called a **constant function**. If the single element in the range is 0, then the function is the **zero function**. Thus, a zero function is a particular case of a constant function. We can briefly describe the zero function from A to B by $f : A \to B : f(a) = 0$.

Before going further we give another definition.
Definition : Two functions f and g, with the same domain A and codomain B, are called **equal** if $f(x) = g(x) \;\forall\; x \in A$.

Now for some exercises.

E 24) a) Define $f : \mathbf{R} \to \mathbf{R}: f(x) = x^2$. Find R_f.

b) Find R_f for $f : \mathbf{R} \to \mathbf{R} : f(x) = 3x$.

E 25) Ten people numbered 1,2,3,...., 10 are classified as right-handed or left-handed. This association gives a function. What is the domain and codomain of this function?

Now that we have defined functions, let us discuss some particular types of functions.

1.7 SOME TYPES OF FUNCTIONS

In this section we shall discuss four types of functions — surjective, injective, bijective and monotonic functions.

1.7.1 Surjective (Onto) Functions

We start with defining a surjective function.

Definition : A function $f : A \to B$ is said to be **surjective** (or **onto**) if the range of f equals its codomain, that is $R_f = B$. In this case, we also say that f is from A **onto** B.

For example, the function in E 24 (b) is surjective.

Given a function $f : A \to B$, how can we verify whether it is surjective or not? One way is to compute R_f and then see whether $R_f = B$ or not. The other way is to use the following criterion:

For each $y \in B$, we should be able to find at least one $x \in A$ such that $y = f(x)$.

In other words, each element y of the codomain B should be the image under f, of at least one element of the domain A.

By this criterion we can say that the function f, defined in Exampl 9, is no surjective. Why? Well, $-10 \in \mathbf{R}$, the codomain of f. But there is no $x \in \mathbf{R}$ for which $x^2 + 1 = -10$.

On the other hand, the function f defined in Example 8 (i) is certainly surjective as, for each $y \in R$, there exists x (= y) such that f(x) = y.

The following exercises will help you in understanding the concept of surjective functions.

E 26) Let A = {1,2,3} and B = {2,4}. Let $f : A \to B$ be defined by f(1) = 2, f(2) = 2, f(3)= 4. Is f surjective?

E 27) Define $f : \mathbf{R} \to \mathbf{R} : f(x) = \frac{x^2}{1 + x^2}$. Is f surjective?

Let us now discuss another type of function.

1.7.2 Injective (One-one) Functions

Another kind of function that you will be coming across is an injective function. Let us define it.

Definition : A function $f : A \to B$ is called **injective** (or **one-one**) if different elements of A have distinct images in B under f, that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, where $x_1 \in A$, $x_2 \in A$.

What does this mean? To clarify this let us consider the identity function $I_A : A \to A : I_A(a) = a$ If $a_1 \neq a_2$, clearly $I_A(a_1) \neq I_A(a_2)$, for $a_1, a_2 \in A$. This function is, therefore, one-one.

We briefly say **f** is **1–1**, if we mean f is one-one.

One way of finding out whether a function f is 1–1 is to show that

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for x_1, x_2 in the domain of f.

Let us look at some examples.

Example 11 : Consider the function $f : \mathbf{R} \to \mathbf{R} : f(x) = \frac{5}{9}(x-32)$. Is it injective?

Solution : It is injective because

$f(x_1) = f(x_2) \Rightarrow \frac{5}{9}(x_1 - 32) = \frac{5}{9}(x_2 - 32) \Rightarrow x_1 = x_2$.

Note that this function is also surjective, since $R_f = \mathbf{R}$.

Did you notice that the function of Example 11 can be written as $f(F) = \frac{5}{9}(F - 32)$? then C = f(F) gives the relationship between the Centigrade and the Fahrenheit scales for measuring temperature.

Example 12 : Consider the absolute value function

$f : \mathbf{R} \to \mathbf{R} : f(x) = |x|$.Is f 1–1?

Solution : By definition $|1| = 1$ and $|-1| = 1$. Therefore, f(1) = f(–1), but $1 \neq (-1)$. Thus, f is not 1–1.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Try the following exercises now.

E 28) Define $f : [0, 2[\to \mathbf{R} : f(x) = \begin{cases} x, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$

Is f surjective? injective?

E 29) Consider the greatest integer function $f : R \to \mathbf{R} : f(x) = [x]$, where [x] is defined to be the integer n if $x \in [n, n + 1[$. (For example, $[1/2] = 0, [2] = 2$ and $[\pi] = 3$.) Why is f not 1 – 1?

And now, we combine the two types of functions that we have done so far.

1.7.3 Bijective Functions

Consider the identity function on A. This is both surjective and injective. Such a function is called bijective. We give the following definition.

Definition : A function $f : A \to B$ is said to be **bijective** if

i) f is surjective, that is, f is from A onto B, and

ii) f is one-one.

A bijective function $f : A \to B$ is also called a **one-to-one correspondence** between A and B.

Note that a function $f : A \to B$ will be bijective only if both (i) and (ii) hold.

Consider the following example.

Example 13: Consider $f : \{1,2,3\} \to \{1,2,3\}$ defined by $f(1) = 2, f(2) = 3, f(3) = 1$. Is f bijective?

Solution : f is bijective as it is both one-one and onto. (f is an example of a **permutation**, that is, a bijection of a set onto itself.)

The following exercise is interesting.

E 30) When can a constant function $f : A \to B$ be bijective?

And now we will study monotonic functions.

1.7.4 Monotonic Functions

Let us again consider the identity function on **R**. As the value of x increases, the value of $I_R(x)$ also increases. Because of this property we say that I_R is a monotonic function. There are four different types of monotonic functions. Let us define them.

Definition : A function $f : A \to \mathbf{R}$, where A is a non-empty subset of R, is said to be

i) increasing if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$, for all $x_1, x_2 \in A$.

ii) non-decreasing if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$, for all $x_1, x_2 \in A$.

iii) decreasing if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$, for all $x_1, x_2 \in A$.

iv) non-increasing if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$, for all $x_1, x_2 \in A$.

f is **monotonic** if it satisfies any one of the four properties (i)–(iv) defined above.

For example, the identity function on **R** is an increasing as well as a decreasing function. For either reason it is a monotonic function.

Let us look at some more examples.

$\mathbf{R}^+$ = set of positive real numbers

Example 14: Consider the function $f : \mathbf{R}^+ \to \mathbf{R}^+ : f(x) = k/x$, where k is a constant. Is it monotonic?

(Note that this relation can be written as $f(v) = k/V$ or $P = k/V$, where $P = f(V)$. This rule represents Boyle's law.)

Solution : f is decreasing because if $x_1, x_2 \in \mathbf{R}^+$ and $x_1 < x_2$, then $k/x_1 > k/x_2$. Therefore, $f(x_1) > f(x_2)$.

Example 15 : Consider the function $f : [-1, \infty[\to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 2x, & \text{if } x \ge 0 \\ 0, & \text{if } -1 \le x < 0 \end{cases}$$

Is this function increasing? monotonic?

Solution : This function is not increasing, since $-1/2 < 0$ but $f(-1/2) = f(0) = 0$. But it is non-decreasing. Why?

Well, if $0 \le x_1 < x_2$, then $2x_1 < 2x_2$, i.e., $f(x_1) < f(x_2)$.
If $-1 \le x_1 < x_2 < 0$ then $f(x_1) = 0 = f(x_2)$.
If $-1 \le x_1 < 0$ and $x_2 > 0$, then $f(x_1) = 0$, $f(x_2) = 2x_2 > 0$, so that $f(x_1) < f(x_2)$.
Thus, $-1 \le x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$. Hence, f is a non-decreasing function on $[-1, \infty[$, Hence, f is monotonic. Note that f is increasing on the interval $[0, \infty[$
Try this exercise now.

E 31) Consider the function $f : [1, 2] \to [5,15] : f(x) = x^2 + 3x + 5$. Is it a monotonic function?

Let us now discuss a way of obtaining new functions from old.

1.8 COMPOSITE OF FUNCTIONS

Let A, B ,C be three sets and $f : A \to B$, $g : B \to C$ be two functions. Then for each $x \in A$, $f(x) \in B$. $\therefore$ $g(f(x)) \in C$. We define a new function, denoted by $g \circ f : A \to C$ as follows:

$(g \circ f)(x) = g(f(x)) \ \forall \ x \in A.$

The function $g \circ f$ is called the **composite** of f and g.
Note : If we want to talk about the composite $g \circ f$, then the range of f must be a subset of the domain of g. For example, if $f : \mathbf{R} \to \mathbf{R}$ is onto and $g : \mathbf{N} \to \mathbf{N}$, then $g \circ f$ is not defined.

Let us look at some examples where the composite is defined.
Example 16 : Consider the functions $f : [1, 2] \to [1, 4] : f(x) = x^2$ and $g : [1, 4] \to [1 ,2] : g(x) = \sqrt{x}$. What is $g \circ f$?
Solution : $g \circ f : [1, 2] \to [1, 2]$ is defined by

$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = x.$

Thus, $g \circ f$ is the identity function on [1,2].

In Example 16, notice that both $g \circ f$ and $f \circ g$ are defined. In this case, $f \circ g : [1, 4] \to [1, 4] : f \circ g\,(x) = x$.
Thus, both $f \circ g$ and $g \circ f$ are identity functions, but $f \circ g = I_{[1,4]}$ and $g \circ f = I_{[1,2]}$.

Now look at another example.
Example 17 : Let $f : [0, 1] \to [5, 6] : f(x) = x + 5$ and $g : [5, 6] \to \mathbf{R} : g(x)=x^2$. Is $g \circ f$ defined?
Solution : $g \circ f$ is well-defined since $R_f \subseteq$ domain of g. In fact, $g \circ f : [0, 1] \to \mathbf{R}$: $g \circ f\,(x) = (x+5)^2$

In Example 17 $f \circ g$ is not defined. Thus, it is not necessary that $f \circ g$ and $g \circ f$ should both be defined. However, sometimes it is possible that both $g \circ f$ and $f \circ g$ are defined. Under such circumstances, a natural question is: Is $g \circ f = f \circ g$? The following example answers this question.
Example 18: Consider $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ defined by

$f(x) = x-1$ and $g(x) = x^2 \ \forall \ x \in \mathbf{R}$. Is $f \circ g = g \circ f$?

Solution : Firstly, note that both $g \circ f$ and $f \circ g$ are defined. Now, $g \circ f\,(x) = g(f(x)) = g(x-1) = (x-1)^2 = x^2-2x + 1$,
whereas $f \circ g\,(x) = f(g(x)) = f(x^2) = x^2-1$.
Thus, $g \circ f\,(x) \neq f \circ g\,(x)$.

Example 18 tells us that the order in which f and g are applied is very important, that is, $f \circ g$ may not be equal to $g \circ f$. This leads us to an important definition.
Definition : If $f : A \to B$ and $g : B \to A$ are such that $f \circ g = I_B$ and $g \circ f = I_A$, then f and g are called the **inverses** of each other. In this case f (and g) are called **invertible** functions.

For example, the function $f : \mathbf{N} \cup \{0\} \to \mathbf{N} : f(x) = x + 1$ and $g : \mathbf{N} \to \mathbf{N} \cup \{0\} : g(x) = x - 1$ are inverses of each other.

There are cases when $f \circ g = g \circ f$, but neither of them are the identity function. For example, if $f : \mathbf{R} \to \mathbf{R} : f(x) = x^2$ and $g = f$, then $f \circ g = g \circ f = f^2 \neq I$.

Try these exercises now.

E 32) Give an example, other than the ones given so far, in which $f \circ g = g \circ f$.

E 33) Write $h : \mathbf{R} \to \mathbf{R} : h(x) = (x^2 + 1)^2$ as a composite of two functions.

Let us now see how to apply the basic arithmetic operations on functions.

1.9 OPERATIONS WITH FUNCTIONS

Just as we add, subtract, multiply and divide numbers to obtain new numbers, we add, subtract, multiply and divide functions to obtain new functions. However, we can do so only under certain restrictions.

Let B be a subset of **R**, and $f : A \to B$ and $g : A \to B$ be two functions. We define the functions $f + g : A \to B$, $f - g : A \to B$ and $fg : A \to B$ as follows:

$(f + g)(x) = f(x) + g(x) \; \forall \; x \in A.$

$(f - g)(x) = f(x) - g(x) \; \forall \; x \in A.$

$(f.g)(x) = f(x) \,.\, g(x) \; \forall \; x \in A.$

Notice that the functions f and g have the same domain A. So the domain of each of the functions $f + g$, $f - g$ and $f.g$ is also the set A.

Difficulties do arise if we want to define f/g. If $f : A \to B$ and $g : A \to B$, then f/g is defined only at those points of A where g does not vanish. In other words, if $x \in A$ is such that $g(x) \neq 0$, then and only then can we define f/g at x. The required definition is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad x \in A, \quad g(x) \neq 0.$$

Similar facts have to be taken into consideration while defining g/f.

Let us look at some examples.

Example 19: Consider $f : [0, \infty[\to \mathbf{R} : f(x) = x^2$, $g : [0, \infty[\to \mathbf{R} : g(x) = \sqrt{x}$. Define f + g, fg and f/g.

Solution: For each $x \in [0, \infty[$

$(f + g)(x) = f(x) + g(x) = x^2 + \sqrt{x}$

$(f.g)(x) = f(x) \,.\, g(x) = x^2\sqrt{x}$

The function g vanishes only at x = 0. Hence the domain of f/g is the set $]0, \infty[$. For each $x \in]0, \infty[$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{\sqrt{x}} = x^{3/2}.$$

Example 20 : Consider $f : \mathbf{R} \to \mathbf{R} : f(x) = 2x$ and $g : \mathbf{R} \to \mathbf{R} : g(x) = x - 1$. Define fg, f/g and g/f.

Solution : The function f.g is defined from **R** to **R**. For each $x \in \mathbf{R}$,

$(f.g)(x) = f(x) \cdot g(x) = 2x(x - 1).$

The function f vanishes only at x = 0. On the other hand, the function g vanishes only at x = 1. Hence, the domain of f/g is $\mathbf{R} \sim \{1\}$, whereas that of g/f is $\mathbf{R} \sim \{0\}$. Thus, the domains of f/g and g/f are different. The definitions of f/g and g/f are

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x}{x-1}, \; x \in \mathbf{R}, \; x \neq 1, \text{ and}$$

$$\left(\frac{f}{g}\right)(x) = \frac{g(x)}{f(x)} = \frac{x-1}{2x}, \; x \in \mathbf{R}, \; x \neq 0.$$

Try this exercise now

E34) Consider the functions $f : \mathbf{R} \to \mathbf{R} : f(x) = x + 5$ and $g : \mathbf{R} \to \mathbf{R} : g(x) = 5 - x$. Define $f + g$, $f - g$, fg, f/g and g/f.

We end the unit with a summary of what we have done in it.

1.10 SUMMARY

In this unit we have covered the following points:

1) The definition of sets as well-defined collections.
2) Different methods of representing sets.
3) The definition of subsets and equality of sets.
4) The use of Venn diagrams to represent sets.
5) The operations of complementation, intersection and union of sets.
6) Statements of the distributive laws and De Morgan's laws.
7) The definition of a function.
8) Some special types of functions such as surjective, injective, bijective and monotonic functions.
9) The definition of the composite of two functions.
10) The sum, difference, product and quotient of functions.

1.11 SOLUTIONS/ANSWERS

E 1) (b), (c) and (d) are sets. But (a) is not a well-defined collection.

E 2) (a) and (c) are true.

E 3) a) $\{2\}$, b) $\{1, 2, 3, 4, 6, 12\}$, c) $\{2, -2\}$, d) $\{8\}$, e) $\{m, a, l, y\}$.

E 4) a) $\{x : x$ is the square of an integer$\}$

b) $\{x : x$ is a prime number$\}$

c) $\{x : x$ is 0 or x is an even integer$\}$

E 5) $\{x : x$ is a real number lying between 1 and 2$\}$

E 6) (a), (c), (d) and (e) are infinite.

(b) and (f) are finite.

E 7) The subsets of $\{1, 2, 3\}$ are

ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1,3\}$, $\{2, 3\}$, $\{1, 2, 3\}$.

a) 1, b) 3, c) 3, d) 1.

E 8) We must show that every element of A belongs to C. Now let $a \in A$. Since $A \subseteq B$, $a \in B$. Since $B \subseteq C$, $a \in C$. Thus, $A \subseteq C$.

E 9) Clearly $A \supseteq C$.

E 10) A square is a rectangle, and a rectangle is a parallelogram. We can take the set of parallelograms as our universal set U. Let S denote the set of squares and R denote the set of rectangles. Then the diagram in Fig. 7 represents this situation.

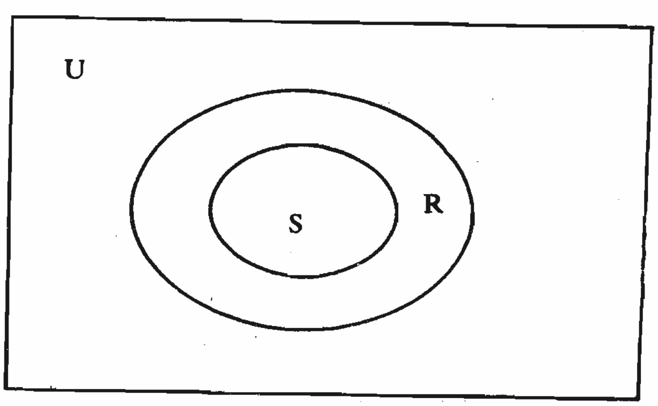

Fig. 7

E 11) Let us take Q as our universal set. The Venn diagram in Fig. 8 represents the given situation.

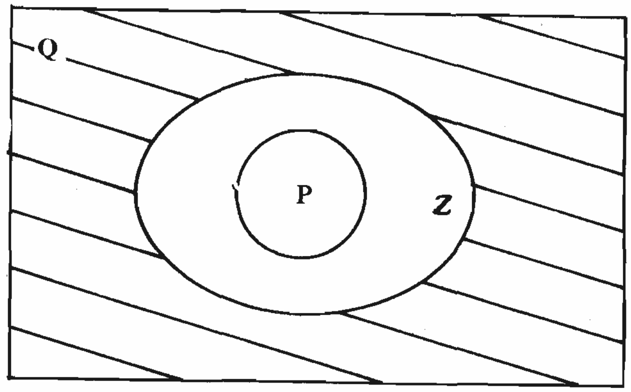

Fig. 8

The shaded portion represents **Q ~ Z**.

Z ~ P is the infinite set $\{0, \pm 1, \pm 4, \pm 6, \pm 8, \pm 9, \ldots\}$

E 12) $A \cap B = \phi$, $B \cap C = \{2\}$, $C \cap D = \phi$, $B \cap D = \phi$.

E 13)

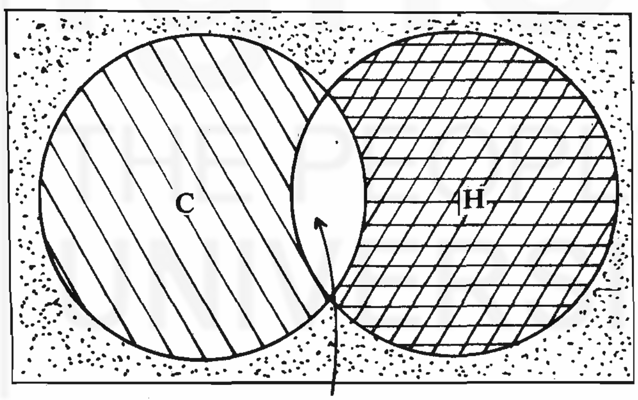

Fig. 9

In Fig. 9 the two circles represent C and H. The single shaded area represents C ~ H. The double shaded area represents H ~ C. The double shaded area and the dotted area together represent C′.

E 14) $A \cap B \cap C = \{4\}$

a) $A \cap B = \{3,4\}$. $\therefore$ $(A \cap B) \cap C = \{4\} = A \cap B \cap C$

b) $B \cap C = \{4\}$. $\therefore$ $A \cap (B \cap C) = \{4\} = A \cap B \cap C$.

'⇔' denotes 'if and only if' or 'implies and is implied by'.

E 15) We need to show that $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$.

Now, $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

$\Leftrightarrow x \in B$ or $x \in A$

$\Leftrightarrow x \in B \cup A$.

$\therefore$ $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$. $\therefore$ $A \cup B = B \cup A$.

E 16) $A \cup B$ is the set of all people who have red hair or black hair. $A \cap B = \phi$.

E 17) $A \cup B$ = set of people suffering from typhoid or cholera or both.

E 18) $A \cup B = \{x \in R : 0 \leq x < 3\}$.

E 19) $A \cup B = \phi \Rightarrow A = \phi$ and $B = \phi$.

E 20) $A \cup B \cup C = \{1, 2, 3, 4, 5\}$

$A \cup B = \{1, 2, 3, 4, 5\}$. $\therefore$ $(A \cup B) \cup C = \{1, 2, 3, 4, 5\} = A \cup B \cup C$.

$B \cup C = \{1, 2, 3, 4, 5\}$. $\therefore$ $A \cup (B \cup C) = \{1, 2, 3, 4, 5\} = A \cup B \cup C$

$= (A \cup B) \cup C$.

E 21) $A \cup B = \{1, 2, 3, 4, 5, 6\}$

$A \sim B = \{1, 2\}$, $A \cap B = \{3\}$, $B \sim A = \{4, 5, 6\}$

$\therefore$ $(A \sim B) \cup (A \cap B) \cup (B \sim A) = \{1, 2, 3, 4, 5, 6\} = A \cup B$.

E 22) a) $B \cup C = \{2, 3, 4, 5\}$ $\therefore$ $A \cap (B \cup C) = \phi$.

Also, $A \cap B = \phi$, $A \cap C = \phi$. So that $(A \cap B) \cup (A \cap C) = \phi = A \cap (B \cup C)$

b) Both sides are equal to $\{1, 3, 4\}$.

E 23) $A \cap B = \phi$. $\therefore$ $(A \cap B)' = U = A' \cup B'$.

$A \cup B = U$. $\therefore$ $(A \cup B)' = \phi = A' \cap B'$.

E 24) a) $R_f = \{x^2 : x \in \mathbf{R}\}$

b) $R_f = \{3x : x \in R\} = \mathbf{R}$, because any $y \in \mathbf{R}$ can be written as $3(y/3) = f(y/3)$.

E 25) Domain = {1,2, ...,10}

Codomain = {right-handed, left-handed}

E 26) Yes.

E 27) No. For example, $-1, \notin R_f$

E 28) Since $2 \notin R_f$, f is not surjective.

Since $f(3/2) = f(6/5) = -1$ and $3/2 \neq 6/5$, we see that f is not injective.

E 29) For example, $f(1) = f(5/4) = 1$, but $1 \neq 5/4$.

E 30) Since f is constant, B is a singleton. Since f is injective, A must also be a singleton. Thus, both A and B will have to be singletons if f is bijective.

E 31) f is an increasing function, since

$x_1 < x_2 \Rightarrow x_1^2 + 3x_1 + 5 < x_2^2 + 3x_2 + 5 \; \forall \; x_1, \; x_2 \in [1, 2,]$.

$\therefore$ f is monotonic.

E 32) We give the following example.

Let $f : \mathbf{R} \to \mathbf{R} : f(x) = x\sqrt{2}$ and $g : \mathbf{R} \to \mathbf{R} : \; g(x) = \frac{x}{\sqrt{2}}$.

Then $g \circ f = f \circ g = I_{\mathbf{R}}$.

E 33) We can write $h = gof$, where $f : \mathbf{R} \to \mathbf{R} : f(x) = x^2 + 1$ and $g : \mathbf{R} \to \mathbf{R} : g(x) = x^2$.

E 34) $f + g$, $f - g$ and fg are functions from $\mathbf{R}$ to $\mathbf{R}$.

Their definitions are

$(f + g)(x) = 10$

$(f - g)(x) = 2x$

$(fg)(x) = 25 - x^2$.

$\forall x \in \mathbf{R}$.

g vanishes at the point $x = 5$ and f vanishes at the point $x = -5$.

$$\therefore \frac{f}{g} : \mathbf{R} \sim \{5\} \to \mathbf{R} : \left(\frac{f}{g}\right)(x) = \frac{5 + x}{5 - x}, \quad \text{and}$$

$$\frac{g}{f} : \mathbf{R} \sim \{-5\} \to \mathbf{R} : \left(\frac{g}{f}\right)(x) = \frac{5 - x}{5 + x}$$

UNIT 2 GRAPHS AND FUNCTIONS

Structure

2.1 INTRODUCTION

In Unit 1 you were introduced to the concept of a function. Sometimes it is easier to understand the way a function behaves if we can view it pictorially. A graph is such a pictorial form. In this unit we will first show you how to draw the graph of any given function. Then we discuss graphs of the power, exponential, logarithmic and trigonometric functions because we expect you to use these functions again and again.

As in the previous unit, the material given in this unit must be studied thoroughly, because it will be useful for the rest of the course.

Objectives

After studying this unit, you should be able to

- draw the graph of a function;
- work with the function $x \to a^x, a > 0, a = 1$;
- define and use logarithmic functions with base 2,10 and e;
- define trigonometric ratios of angles and trigonometric functions of real numbers.

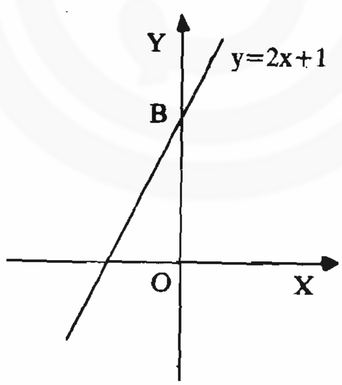

Fig. 1 : Graph of y = 2x+1

2.2 GRAPHS

Consider the function $f : \mathbf{R} \to \mathbf{R} : f(x) = 2x + 1$. Will you believe that its pictorial representation is given by the line AB in Fig. 1? You will, by the time you get to the end of this section.

To draw a graph we need to know about the Cartesian coordinate system. So let us talk about Cartesian coordinates now.

2.2.1 Coordinates

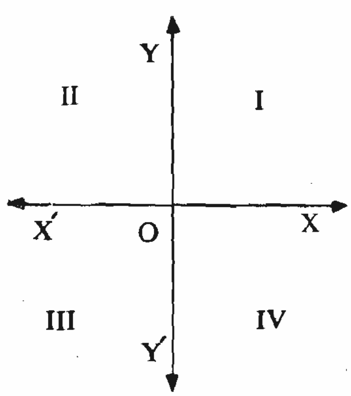

Fig. 2 : The 4 quadrants

The French mathematician and philosopher, Rene Descartes, was the first to suggest a system to represent functions pictorially. It is called the Cartesian system. The basic idea behind this system is to locate points in a plane. For this, we take any point O in a plane and draw two lines passing through it and at right angles to each other (see Fig. 2). Let us name these lines X′OX and Y′OY. X′OX is horizontal, whereas Y′OY is vertical. The point O is called the **origin**. The lines X′OX and Y′OY are called **the x-axis** and **the y-axis,** respectively. OX and OY are the positive directions of the x and y axes, respectively. OX′ and OY′ are the negative directions of the x and y axis, respectively. That is, the distances of points on OX and OY from O are taken as positive. The distances of points of OX′ and OY′ from O are taken as negative.

The x-axis and the y-axis divide the whole plane into four regions, namely, the 1st, 2nd, 3rd and 4th **quadrants** (see Fig. 2).